AN EQUATION CONCERNING THE SMARANDACHE LCM FUNCTION

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Abstract: In this paper we completely solve an open problem concerning the Smarandache LCM function.

Key words: Smarandache function; Smarandache LMC function; diophantine equation

For any positive integer n, let $S(n)$ be the Smarandache function. For any positive integer k, let $L(k)$ be the least common multiple of $1,2,\cdots,k$. Further, let $SL(n)$ denote the least positive integer k such that $L(k)\equiv 0 \pmod{n}$. Then $SL(n)$ is called the Smarandache LCM function. In [2], Murthy' showed that if n is a prime, the $SL(n)=S(n)=n$. Simultaneously, he proposed the following problem.

$$SL(n)=S(n),\ S(n)\neq n? \tag{1}$$

Supported by the National Natural Science Foundation of China (No.10271104), the Guangdong Provincial Natural Science Foundation (No.011781) and the Natural Science Foundation of the Education Department of Guangdong Province (No.0161).

In this paper we completely solve the above mentioned problem as follows:

Theorem. Every positive integer n satisfying (1) can be expressed as

$$n=12 \quad \text{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p, \tag{2}$$

where $p_1, p_2, \cdots, p_r$, p are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$ $(i=1,2,\cdots,r)$.

The above theorem means that (1) has infinitely many positive integer solutions n. The proof of our theorem depends on the following lemmas.

Lemma 1 ([1]). Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \tag{3}$$

be the factorization of n. Then we have

$$S(n) = \max\left(S\left(p_1^{\alpha_1}\right), S\left(p_2^{\alpha_2}\right), \cdots, S\left(p_t^{\alpha_t}\right)\right).$$

Lemma 2 ([1]). If p^α is a power of prime, then $S\left(p^\alpha\right) \equiv 0 \pmod{p}$.

Lemma 3 ([1]). If p^α is a power of prime such that $\alpha > 1$ and $p^\alpha \neq 4$, then $S\left(p^\alpha\right) < p^\alpha$.

Lemma 4 ([2]). If (3) is the factorization of n, then $SL(n)=\max\left(p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_t^{\alpha_t}\right)$.

Proof of Theorem. Let n be a positive integer solution of (1). Further, let (3) be the factorization of n, and let

$$p^\alpha = \max\left(p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_t^{\alpha_t}\right). \tag{4}$$

By Lemmas 1 and 4, we get from (1), (3) and (4) that

$$p^\alpha = SL(n) = S(n) = S(p_j^{\alpha_j}), \qquad 1 \le j \le t. \tag{5}$$

By Lemma 2, we have $S(p_j^{\alpha_j}) \equiv 0 \pmod{p_j}$. Hence, by (5), we get $p=p_j$ and

$$p^\alpha = S\left(p^\alpha\right). \tag{6}$$

If $p^\alpha = 4$, then from (4) we get $n=4$ or 12.

Since $S(4)=S(12)=4$ and $S(n) \neq n$, we obtin $n=12$.

If $\alpha=1$, then from (4) we get $j=t$. Since $S(n) \neq n$, we see from (3) that $t>1$. Let $r=t-1$. Then, by (3), we obtain (2). Thus, the theorem is proved.

References

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