An inequality between prime powers dividing $n!$

Florian Luca

For any positive integer $n \geq 1$ and for any prime number p let $e_p(n)$ be the exponent at which the prime p appears in the prime factor decomposition of $n!$. In this note we prove the following:

Theorem.

Let $p < q$ be two prime numbers, and let $n > 1$ be a positive integer such that $pq \mid n$. Then,

$$p^{e_p(n)} > q^{e_q(n)}. \tag{1}$$

Inequality (1) was suggested by Balacenoiu at the First International Conference on Smarandache Notions in Number Theory (see [1]). In fact, in [1], Balacenoiu showed that (1) holds for $p = 2$. In what follows we assume that $p \geq 3$.

We begin with the following lemmas:

Lemma 1.

(i) *The function*

$$f(x) = \frac{x-1}{\log x} \tag{2}$$

is increasing for $x \geq e$.

(ii) *Let $p \geq 3$ be a real number. Then,*

$$x > (p-1)\log_p(x) \qquad \text{for } x \geq p. \tag{3}$$

(iii) *Let $p \geq 3$ be a real number. The function*

$$g_p(x) = \frac{x-2}{x-(p-1)\log_p(x)} \tag{4}$$

is positive and decreasing for $x \geq p(p+2)$.

(iv)

$$\frac{p+2}{p} > \frac{\log(p+4)}{\log p} \qquad \text{for } p > e^2. \tag{5}$$

(v)

$$\frac{p+1}{p} > \frac{\log(p+2)}{\log p} \qquad \text{for } p > e. \tag{6}$$

Proof. (i) Notice that

$$\frac{df}{dx} = \frac{1}{\log^2 x} \cdot \left(\log\left(\frac{x}{e}\right) + \left(\frac{1}{x}\right)\right) > 0 \qquad \text{for } x > e.$$

(ii) Suppose that $x \geq p \geq 3$. From (i) it follows that

$$\frac{x}{\log x} > \frac{x-1}{\log x} \geq \frac{p-1}{\log p}. \tag{7}$$

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Inequality (7) is clearly equivalent to

$$x > (p-1)\frac{\log x}{\log p} = (p-1)\log_p(x).$$

(*iii*) The fact that $g_p(x) > 0$ for $x \geq p \geq 3$ follows from (*ii*). Suppose that $x \geq p(p+2)$, and that $p \geq 3$. Then,

$$\frac{dg_p}{dx} = \frac{-\log(p)\big((p-1)x\log x - (2\log p + p - 1)x + 2(p-1)\big)}{x\big((p-1)\log x - x\log p\big)^2}. \tag{8}$$

From (8), it follows that in order to check that $dg_p/dx < 0$ it suffices to show that

$$(p-1)x\log x - (2\log p + p - 1)x > 0,$$

or that

$$\log x > \Big(2\frac{\log p}{p-1} + 1\Big) = \Big(\frac{2}{f(p)} + 1\Big). \tag{9}$$

The left hand side of (9) is increasing in x. By (*i*), the right hand side of (9) is decreasing in p. Thus, since $p \geq 3$, and $x \geq p(p+2) \geq 15$, it suffices to show that inequality (9) holds for $x = 15$ and $p = 3$. But this is straightforward.

(*iv*) Inequality (5) is equivalent to

$$p^{p+2} > (p+4)^p,$$

or

$$p^2 > \Big(1 + \frac{4}{p}\Big)^p = \Big[\Big(1 + \frac{4}{p}\Big)^{p/4}\Big]^4. \tag{10}$$

Since

$$e > (1+x)^{1/x} \qquad \text{for all } x > 0, \tag{11}$$

it follows, from inequality (11) with $x = 4/p$, that

$$e > \Big(1 + \frac{4}{p}\Big)^{p/4}. \tag{12}$$

From inequality (12) one can immediately see that (10) holds whenever $p > e^2$.

(*v*) Follows from arguments similar to the ones used at (*iv*).

For every prime number p and every positive integer n let $\tau_p(n)$ be the sum of the digits of n written in the base p.

Lemma 2.

Let $p < q$ be two prime numbers and let n be a positive integer. Assume that $pq \mid n$. Then,

(*i*) $\tau_q(n) \geq 2$.

(*ii*) $\tau_p(n) < (p-1)\log_p(n)$.

Proof. (*i*) Since $n > 0$ it follows that $\tau_q(n) \geq 1$. If $\tau_q(n) = 1$, it follows that n is a power of q which contradicts the fact that $p \mid n$. Hence, $\tau_q(n) \geq 2$.

(*ii*) Let $n = pql$ for some integer $l \geq 1$. Let

$$ql = a_0 + a_1p + \ldots + a_sp^s,$$

where $0 \le a_i \le p-1$ for $1 \le i \le s$, and $a_s \neq 0$, be the representation of ql in the base p. Clearly,

$$s = [\log_p(ql)] < \log_p(ql).$$

Since

$$n = pql = a_0 p + a_1 p^2 + \ldots + a_s p^{s+1},$$

it follows that

$$\tau_p(n) = \sum_{i=0}^{s} a_i \le (p-1)(s+1) < (p-1)(\log_p(ql)+1) = (p-1)\log_p(n).$$

The Proof of the Theorem. Suppose that $q > p \ge 3$ are prime numbers, and that $n > 1$ is such that $pq \mid n$. By applying logarithms in (1) it suffices to prove that

$$e_p(n)\log p > e_q(n)\log q. \tag{13}$$

Since

$$e_p(n) = \frac{n-\tau_p(n)}{p-1} \text{ and } e_q(n) = \frac{q-\tau_q(n)}{q-1},$$

it follows that (13) can be rewritten as

$$\frac{n-\tau_p(n)}{p-1}\cdot \log p > \frac{n-\tau_q(n)}{q-1}\cdot \log q,$$

or

$$\frac{(q-1)\log p}{(p-1)\log q} > \frac{n-\tau_q(n)}{n-\tau_p(n)}. \tag{14}$$

We distinguish two cases:

CASE 1. $q = p+2$. We distinguish two subcases:

CASE 1.1. $n = pq$. In this case, since $q = p+2$, and $p \ge 3$, it follows that $\tau_p(n) = \tau_p(p^2+2p) = 3$, and $\tau_q(n) = \tau_q(pq) = p$. Therefore inequality (14) becomes

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{p^2+2p-p}{p^2+2p-3} = \frac{p(p+1)}{p^2+2p-3}. \tag{15}$$

Inequality (15) is equivalent to

$$\frac{p^2+2p-3}{p(p-1)} > \frac{\log(p+2)}{\log p}. \tag{16}$$

By lemma 1 (v) we conclude that in order to prove inequality (16) it suffices to show that

$$\frac{p^2+2p-3}{p(p-1)} \ge \frac{p+1}{p}. \tag{17}$$

But (17) is equivalent to

$$\frac{p^2+2p-3}{p-1} \ge p+1, \tag{18}$$

or $p^2+2p-3 \ge p^2-1$, or $p \ge 1$ which is certainly true. This disposes of Case 1.1.

CASE 1.2. $n = pql$ where $l \geq 2$. In this case $n \geq 2p(p+2) > 2p^2$. By lemma 2 (i) and (ii), it follows that

$$\frac{n-2}{n-(p-1)\log_p(n)} > \frac{n-\tau_q(n)}{n-\tau_p(n)}. \tag{19}$$

Thus, in order to prove (14) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{n-2}{n-(p-1)\log_p(n)} = g_p(n). \tag{20}$$

Since $n > 2p^2 > p(p+2)$, and since $g_p(n)$ is decreasing for $n > p(p+2)$ (thanks to lemma 1 (iii)), it follows that in order to prove (20) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > g_p(2p^2) = \frac{2p^2-2}{2p^2-\log_p(2p^2)}. \tag{21}$$

Since $p \geq 3 > 2^{3/2}$, it follows that $p^{2/3} > 2$. Hence,

$$\log_p(2p^2) < \log_p(p^{2/3}p^2) = \frac{8}{3}.$$

We conclude that in order to prove (21) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{2p^2-2}{2p^2-\frac{8}{3}} = \frac{3(p-1)(p+1)}{3p^2-4}. \tag{22}$$

Inequality (22) is equivalent to

$$\frac{3p^2-4}{3(p-1)^2} > \frac{\log(p+2)}{\log p}. \tag{23}$$

Using inequality (6), it follows that in order to prove (23) it suffices to show that

$$\frac{3p^2-4}{3(p-1)^2} > \frac{p+1}{p}. \tag{24}$$

Notice now that (24) is equivalent to

$$3p^3-4p > 3(p-1)^2(p+1) = 3p^3-3p^2-3p+3,$$

or $3p^2 > p+3$ which is certainly true for $p \geq 3$. This disposes of Case 1.2.

CASE 2. $q \geq p+4$. Using inequality (19) it follows that in order to prove inequality (14) it suffices to show that

$$f(q)\cdot\frac{\log p}{p-1} = \frac{(q-1)\log p}{(p-1)\log q} > \frac{n-2}{n-(p-1)\log_p(n)} = g_p(n). \tag{25}$$

Since $f(q)$ is increasing for $q \geq 3$ (thanks to lemma 1 (i)), and since $g_p(n)$ is decreasing for $n \geq pq \geq p(p+4) > p(p+2)$, it follows that in order to prove (25) it suffices to show that inequality (25) holds for $q = p + 4$, and $n = pq = p(p + 4)$. Hence, we have to show that

$$\frac{(p+3)\log p}{(p-1)\log(p+4)} > \frac{p^2+4p-2}{p^2+4p-(p-1)\log_p\big(p(p+4)\big)}. \tag{26}$$

Inequality (26) is equivalent to

$$\frac{(p+3)}{(p-1)\log(p+4)} > \frac{p^2+4p-2}{(p^2+3p+1)\log p-(p-1)\log(p+4)},$$

or

$$\frac{(p+3)(p^2+3p+1)}{(p-1)(p^2+4p-2)+(p-1)(p+3)} > \frac{\log(p+4)}{\log p},$$

or

$$\frac{p^3+6p^2+10p+3}{p^3+4p^2-4p-1} > \frac{\log(p+4)}{\log p}. \tag{27}$$

One can easily check that (27) is true for $p = 3,\ 5,\ 7$. Suppose now that $p \geq 11 > e^2$. By lemma 1 (*iv*), it follows that in order to prove (27) it suffices to show that

$$\frac{p^3+6p^2+10p+3}{p^3+4p^2-4p-1} > \frac{p+2}{p}. \tag{28}$$

Notice that (28) is equivalent to

$$p^4+6p^3+10p^2+3p > (p+2)(p^3+4p^2-4p-1) = p^4+6p^3+4p^2-9p-2,$$

or $6p^2 + 11p + 2 > 0$, which is obvious. This disposes of the last case.

Reference

[1] I. BALACENOIU, ***Remarkable Inequalities***, to appear in the Proceedings of the First International Conference on Smarandache Type Notions in Number Theory, Craiova, Romania, 1997.

Florian Luca
Department of Mathematics
Syracuse University
215 Carnegie Hall
Syracuse, New York
e-mail: fgluca@syr.edu