An introduction to the Smarandache Double factorial function

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In [1], [2] and [3] the Smarandache Double factorial function is defined as:

Sdf(n) is the smallest number such that Sdf(n)!! is divisible by n, where the double factorial is given by [4]:

m!! = 1x3x5x...m, if m is odd;
m!! = 2x4x6x...m, if m is even.

In this paper we will study this function and several examples, theorems, conjectures and problems will be presented. The behaviour of this function is similar to the other Smarandache functions introduced in the chapter I.
In the table below the first 100 values of fucntion Sdf(n) are given:

n	Sdf(n)	n	Sdf(n)	n	Sdf(n)	n	Sdf(n)	n	Sdf(n)
1	1	21	7	41	41	61	61	81	15
2	2	22	22	42	14	62	62	82	82
3	3	23	23	43	43	63	9	83	83
4	4	24	6	44	22	64	8	84	14
5	5	25	15	45	9	65	13	85	17
6	6	26	26	46	46	66	22	86	86
7	7	27	9	47	47	67	67	87	29
8	4	28	14	48	6	68	34	88	22
9	9	29	29	49	21	69	23	89	89
10	10	30	10	50	20	70	14	90	12
11	11	31	31	51	17	71	71	91	13
12	6	32	8	52	26	72	12	92	46
13	13	33	11	53	53	73	73	93	31
14	14	34	34	54	18	74	74	94	94
15	5	35	7	55	11	75	15	95	19
16	6	36	12	56	14	76	38	96	8
17	17	37	37	57	19	77	11	97	97
18	12	38	38	58	58	78	26	98	28
19	19	39	13	59	59	79	79	99	11
20	10	40	10	60	10	80	10	100	20

According to the experimental data the following two conjectures can be formulated:

Conjecture 4.1 The series $\sum_{n=1}^{\infty} Sdf(n)$ is asymptotically equal to $a \cdot n^b$ where a and b are close to 0.8834.. and 1.759.. respectively.

Conjecture 4.2 The series $\sum_{n=1}^{\infty} \frac{1}{Sdf(n)}$ is asymptotically equal to $a \cdot n^b$ where a and b are close to 0.9411.. and 0.49.. respectively.

Let's start now with the proof of some theorems.

Theorem 4.1. Sdf(p)=p where p is any prime number.

Proof. For p=2, of course Sdf(2)=2. For p odd instead observes that only for m=p the factorial of first m odd integers is a multiple of p, that is $1 \cdot 3 \cdot 5 \cdot 7 \ldots\ldots p = (p-2)!! \cdot p$.

Theorem 4.2. For any squarefree even number n,

$$Sdf(n) = 2 \cdot \max\{p_1, p_2, p_3, \ldots\ldots p_k\}$$

where $p_1, p_2, p_3, \ldots\ldots p_k$ are the prime factors of n.

Proof. Without loss of generality let's suppose that $n = p_1, p_2, p_3$ where $p_3 > p_2 > p_1$ and $p_1 = 2$. Given that the factorial of even integers must be a multiple of n of course the smallest integer m such that $2 \cdot 4 \cdot 6 \ldots \cdot m$ is divisible by n is $2 \cdot p_3$. Infact for $m = 2 \cdot p_3$ we have :

$$2 \cdot 4 \cdot 6 \ldots 2 \cdot p_2 \ldots \cdot 2 \cdot p_3 = (2 \cdot p_2 \cdot p_3) \cdot (4 \cdot 6 \ldots 2 \ldots \cdot 2) = k \cdot (2 \cdot p_2 \cdot p_3) \quad where\ k \in N$$

Theorem 4.3. For any squarefree composite odd number n, $Sdf(n) = \max\{p_1, p_2, \ldots p_k\}$ where $p_1, p_2, \ldots p_k$ are the prime factors of n.

Proof. Without loss of generality let suppose that $n = p_1 \cdot p_2$ where p_1 and p_2 are two distinct primes and $p_2 > p_1$. Of course the factorial of odd integers up to p_2 is a multiple of n because being $p_1 < p_2$ the factorial will contain the product $p_1 \cdot p_2$ and therefore n: $1 \cdot 3 \cdot 5 K \cdot p_1 K \cdot p_2$

Theorem 4.4. $\sum_{n=1}^{\infty} \frac{1}{Sdf(n)}$ diverges.

Proof. This theorem is a direct consequence of the divergence of sum $\sum_{p} \frac{1}{p}$ where p is any prime number.

In fact $\sum_{k=1}^{\infty} \frac{1}{Sdf(k)} > \sum_{p=2}^{\infty} \frac{1}{p}$ according to the theorem 4.1 and this proves the theorem.

Theorem 4.5 The Sdf(n) function is not additive that is $Sdf(n+m) \neq Sdf(n) + Sdf(m)$ *for* $(n,m) = 1$.

Proof. In fact for example $Sdf(2+15) \neq Sdf(2) + Sdf(15)$.

Theorem 4.6 The Sdf(n) function is not multiplicative, that is $Sdf(n \cdot m) \neq Sdf(n) \cdot Sdf(m)$ *for* $(n,m) = 1$.

Proof. In fact for example $Sdf(3 \cdot 4) \neq Sdf(3) \cdot Sdf(4)$.

Theorem 4.7 $Sdf(n) \leq n$

Proof. If n is a squarefree number then based on theorems 4.1, 4.2 and 4.3 $Sdf(n) \leq n$. Let's now consider the case when n is not a squarefree number. Of course the maximum value of the Sdf(n) function cannot be larger than n because when we arrive in the factorial to n for shure it is a multiple of n.

Theorem 4.8 $\sum_{n=1}^{\infty} \frac{Sdf(n)}{n}$ diverges.

Proof. In fact $\sum_{k=1}^{\infty} \frac{Sdf(k)}{k} > \sum_{p=2}^{\infty} \frac{Sdf(p)}{p}$ where p is any prime number and of course $\sum_{p} \frac{Sdf(p)}{p}$ diverges because the number of primes is infinite [5] and Sdf(p)=p.

Theorem 4.9 $Sdf(n) \geq 1$ *for* $n \geq 1$

Proof. This theorem is a direct consequence of the Sdf(n) function definition. In fact for n=1, the smallest m such that 1 divide Sdf(1) is trivially 1. For $n \neq 1$, m must be greater than 1 becuase the factorial of 1 cannot be a multiple of n.

Theorem 4.10 $0 < \frac{Sdf(n)}{n} \leq 1$ for $n \geq 1$

Poof. The theorem is a direct consequence of theorem 4.7 and 4.9.

Theorem 4. 11 $Sdf(p_k\#) = 2 \cdot p_k$ where $p_k\#$ is the product of first k primes (primorial) [4].

Proof. The theorem is a direct consequence of theorem 4.2.

Theorem 4.12 The equation $\frac{Sdf(n)}{n} = 1$ has an infinite number of solutions.

Proof. The theorem is a direct consequence of theorem 4.1 and the well-known fact that there is an infinite number of prime numbers [5].

Theorem 4.13 The even (odd respectively) numbers are invariant under the application of Sdf function, namely Sdf(even)=even and Sdf(odd)=odd

Proof. Of course this theorem is a direct consequence of the Sdf(n) function definition.

Theorem 4.14 The diophantine equation $Sdf(n) = Sdf(n+1)$ doesn't admit solutions.

Proof. In fact according to the previous theorem if n is even (odd respectively) then Sdf(n) also is even (odd respectively). Therefore the equation Sdf(n)=Sdf(n+1) can not be satisfied because Sdf(n) that is even should be equal to Sdf(n+1) that instead is odd.

Conjecture 4.3 The function $\frac{Sdf(n)}{n}$ is not distributed uniformly in the interval]0,1].

Conjecture 4.4 For any arbitrary real number $\varepsilon > 0$, there is some number $n \geq 1$ such that $\frac{Sdf(n)}{n} < \varepsilon$

Let's now start with some problems related to the Sdf(n) function.

Problem 1. *Use the notation FSdf(n)=m to denote, as already done for the Zt(n) and Zw(n) functions, that m is the number of different integers k such that Zw(k)=n.*

Example FSdf(1)=1 since Sdf(1)=1 and there are no other numbers n such that Sdf(n)=1

Study the function Fsdf(n).

Evaluate $\lim_{m \to \infty} \frac{\sum_{k=1}^{m} \frac{FSdf(k)}{k}}{m}$

Problem 2. *Is the difference |Sdf(n+1)-Sdf(n)| bounded or unbounded?*

Problem 3. *Find the solutions of the equations:* $\frac{Sdf(n+1)}{Sdf(n)} = k$ $\frac{Sdf(n)}{Sdf(n+1)} = k$ *where k is any positive integer and n>1 for the first equation.*

Conjecture 4.5 The previous equations don't admits solutions.

Problem 4. *Analyze the iteration of Sdf(n) for all values of n. For iteration we intend the repeated application of Sdf(n). For example the k-th iteration of Sdf(n) is:*

$$Sdf^{k}(n) = Sdf(Sdf(\ldots(Sdf(n))\ldots)$$ *where Sdf is repeated k times.*

For all values of n, will each iteration of Sdf(n) produces always a fixed point or a cycle?

Problem 5. *Find the smallest k such that between Sdf(n) and Sdf(k+n), for n>1, there is at least a prime.*

Problem 6. *Is the number 0.1232567491011.... where the sequence of digits is Sdf(n) for* $n \geq 1$ *an irrational or trascendental number? (We call this number the Pseudo-Smarandache-Double Factorial constant).*

Problem 7. *Is the Smarandache Euler-Mascheroni sum (see chapter II for definition) convergent for Sdf(n) numbers? If yes evaluate the convergence value.*

Problem 8. *Evaluate* $\sum_{k=1}^{\infty}(-1)^k \cdot Sdf(k)^{-1}$

Problem 9. *Evaluate* $\prod_{n=1}^{\infty}\frac{1}{Sdf(n)}$

Problem 10. *Evaluate* $\lim_{k\to\infty}\frac{Sdf(k)}{\theta(k)}$ *where* $\theta(k)=\sum_{n\le k}\ln(Sdf(n))$

Problem 11. *Are there m, n, k non-null positive integers for which* $Sdf(n\cdot m)=m^k\cdot Sdf(n)$?

Problem 12. *Are there integers k>1 and n>1 such that* $(Sdf(n))^k = k\cdot Sdf(n\cdot k)$?

Problem 13. *Solve the problems from 1 up to 6 already formulated for the Zw(n) function also for the Sdf(n) function.*

Problem 14. *Find all the solution of the equation Sdf(n)!=Sdf(n!)*

Problem 15. *Find all the solutions of the equation* $Sdf(n^k)=k\cdot Sdf(n)$ *for k>1and n>1.*

Problem 16. *Find all the solutions of the equation* $Sdf(n^k)=n\cdot Sdf(k)$ *for k>1.*

Problem 17. *Find all the solutions of the equation* $Sdf(n^k) = n^m \cdot Sdf(m)$ *where* *k>1 and n, m >0.*

Problem 18. *For the first values of the Sdf(n) function the following inequality is true:*

$$\frac{n}{Sdf(n)} \leq \frac{1}{8} \cdot n + 2 \quad for\ 1 \leq n \leq 1000$$

Is this still true for n>1000 ?

Problem 19. *For the first values of the Sdf(n) function the following inequality is true:*

$$\frac{Sdf(n)}{n} \leq \frac{1}{n^{0.73}} \quad for\ 1 \leq n \leq 1000$$

Is this still true for all values of n>1000 ?

Problem 20. *For the first values of the Sdf(n) function the following inequality hold:*

$$\frac{1}{n} + \frac{1}{Sdf(n)} < n^{-\frac{1}{4}} \quad for\ 2 < n \leq 1000$$

Is this still true for n>1000 ?

Problem 21. *For the first values of the Sdf(n) function the following inequality holds:*

$$\frac{1}{n \cdot Sdf(n)} < n^{-\frac{5}{4}} \quad for\ 1 \leq n \leq 1000$$

Is this inequality still true for n>1000 ?

Problem 22. *Study the convergence of the Smarandache Double factorial harmonic series:*

$$\sum_{n=1}^{\infty} \frac{1}{Sdf^{a}(n)} \quad \text{where } a > 0 \text{ and } a \in R$$

Problem 23. *Study the convergence of the series:*

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Sdf(x_n)}$$

where x_n *is any increasing sequence such that* $\lim_{n \to \infty} x_n = \infty$

Problem 24. *Evaluate*

$$\lim_{n \to \infty} \frac{\sum_{k=2}^{n} \frac{\ln(Sdf(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

Problem 25. *Solve the functional equation:*

$$Sdf(n)^{r} + Sdf(n)^{r-1} + \Lambda\ \Lambda\ Sdf(n) = n$$

where r is an integer ≥ 2 .
Wath about the functional equation:

$$Sdf(n)^{r} + Sdf(n)^{r-1} + \Lambda\ \Lambda\ Sdf(n) = k \cdot n$$

where r and k are two integers ≥ 2.

Problem 26. *Is there any relationship between* $Sdf\left(\prod_{k=1}^{m} m_k\right)$ *and* $\sum_{k=1}^{m} Sdf(m_k)$?

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