CONVERGENCE OF THE SMARANDACHE GENERAL CONTINUED FRACTION

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ABSTRACT. We give a positive answer to the Smarandache General Continued Fraction convergence (see [2]).

The Smarandache General Continued Fraction associated with the Smarandache reverse sequence 1, 21, 321, 4321, 54321, ..., 121110987654321, ..., is given by

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\dots}}}}. \tag{1}$$

(See for more details [2]).
Define the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ by :

$$a_0 = 1, a_1 = 12, a_2 = 123, ...$$
$$b_0 = 1, b_1 = 21, b_2 = 321, ...$$

We verify easily that

$$a_{n+1} = 10a_n + (n+1) \text{ and } b_{n+1} = 10b_n + \frac{10^{(n+1)}-1}{9}, \text{ for any } n \geq 0. \tag{2}$$

With notations of [1], the continued fraction (1) can be written as follows:

$$a_0 + \frac{b_0|}{|a_1} + \frac{b_1|}{|a_2} + \frac{b_2|}{|a_3} + ... + \frac{b_{n-1}|}{|a_n} + ...$$

Let $\frac{A_k}{B_k}$ be the result of the k^{th} reduce of continued fraction:

$$a_0 + \frac{b_0|}{|a_1} + \frac{b_1|}{|a_2} + \frac{b_2|}{|a_3} + ... + \frac{b_{k-1}|}{|a_k}. \tag{3}$$

Thus we define two sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ of real numbers. Using the elementary algebraic theory of continued fraction given by Euler (see [1]) we have the following,

Lemma 0.1. *The sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ satisfy the following statements:*

$$A_n = a_nA_{n-1} + b_nA_{n-2}, \text{ for } n \geq 2, A_{-1} = 0 \text{ and } A_0 = a_1.$$
$$B_n = a_nB_{n-1} + b_nB_{n-2}, \text{ for } n \geq 2, B_{-1} = 0 \text{ and } B_0 = 1.$$

In consequence, we have:

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{(n-1)}b_0b_1...b_{n-1}, \text{ for any } n \geq 1$$

And if $B_n \neq 0$, for any $n \geq 0$, we have,

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{(n-1)} \frac{b_1 b_2 ... b_{n-1}}{B_{n-1} B_n}, \text{ for any } n \geq 1$$

An easy computation gives,

$$\frac{A_n}{B_n} = a_0 + \sum_{k=1}^{n} (-1)^{(k-1)} \frac{b_0 b_1 ... b_{k-1}}{B_{k+1} B_k}$$

Hence, we have the following result

Lemma 0.2. *The Smarandache General Continued Fraction (1) is convergent if and only if the alternate series* $\sum_{k=1}^{n} (-1)^{(k-1)} \frac{b_0 b_1 ... b_{k-1}}{B_{k-1} B_k}$ *is also convergent.*

Let $\{u_n\}_{n \geq 0}$ be the sequence of positive real numbers defined by

$$u_n := \frac{b_0 b_1 ... b_{n-1}}{B_{n-1} B_n} \text{ for } n \geq 1$$

We have,

$$\begin{aligned} u_{n+1} - u_n &= \frac{b_0 b_1 ... b_n}{B_n B_{n+1}} - \frac{b_0 b_1 ... b_{n-1}}{B_{n-1} B_n}. \\ &= \frac{b_0 b_1 ... b_{n-1}}{B_n} [\frac{b_n}{B_{n+1}} - \frac{1}{B_{n-1}}]. \\ &= \frac{b_0 b_1 ... b_{n-1}}{B_{n-1}} [\frac{b_n B_{n-1} - B_{n+1}}{B_{n-1} B_{n+1}}]. \end{aligned}$$

And using the lemma 0.1 , we get

$$u_{n+1} - u_n = \frac{b_0 b_1 ... bn - 1}{B_{n-1} B_n} [\frac{b_n - b_{n+1}(B_{n-1} - a_{n+1} B_n)}{B_{n-1} B_{n+1}}]$$

And by (2) we have

$$b_n - b_{n+1} = -9 b_n - \frac{10^{(n+1)} \quad 1}{9} \leq 0$$

Because the B_n's are positive, we deduce that the sequence $\{u_n\}_{n \geq 0}$ is decreasing. On the other hand, we have, $B_n B_{n\ 1} = [a_n B_{n-1} + b_n B_{n-2}] B_{n-1} \geq b_2 b_3 ... b_n B_1 B_0$, which implies that

$$u_n \leq \frac{b_0 b_1 ... b_{n\ 1}}{b_2 b_3 ... b_n B_1 B_0} = \frac{b_1}{b_n B_1} = \frac{1}{b_n}.$$

The last inequality assert that $\lim_{n \to +\infty} u_n = 0$. Finally, we have the result

Theorem 0.1. *The Smarandache General Continued Fraction (1) is convergent.*

References

[1] J.Dieudonne, *Fraction continues et polynomes orthogonaux dans l'oeuvre de E.N.Laguerre*, Proceeding of the Laguerre symposium held at Bar-le-Duc, October 15-18, (1984).

[2] Castillo, Jose, *Smarandache Continued Fractions*, Bulletin of Pure and applied Sciences, Delhi, India, Vol.17E, N1, 149-151, 1988.