DECOMPOSITION OF THE DIVISORS OF A NATURAL NUMBER INTO PAIRWISE CO-PRIME SETS

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Given n a natural number . Let $d_1, d_2, d_3, d_4, d_5, \ldots$ be the divisors of N. A querry coms to my mind, as to, in how many ways , we could choose a divisor pair which are co-prime to each other? Similarly in how many ways one could choose a triplet, or a set of four divisors etc. such that, in each chosen set, the divisors are pairwise co-prime.?

We start with an example Let $N= 48 = 2^4 \times 3$. The ten divisors are

1 , 2 , 3, 4, 6, 8, 12, 16, 24, 48

We denote set of co-prime pairs by $D_2(48)$, co-prime triplets by $D_3(48)$ etc.

We get $D_2(48) = \{ (1,2) , (1, 3) , (1 , 4), (1 , 6), (1,8), (1, 12), (1,16) , (1 , 24), (1 , 48),$

$(2, 3), (4,3), (8, 3) , (16, 3) \}$

Order of $D_2(48) = 13$.

$D_3(48) = \{ (1,2, 3) , (1, 3, 4) , (1 , 3 , 8), (1 ,3, 16) \}$, **Order of $D_3(48) = 4$.**

$D_4(48) = \{ \} = D_5(48) = \ldots = D_9(48) = D_{10}(48)$.

Another example N = 30 = 2x3x5 (a square free number). The 8 divisors are

1 , 2 , 3 , 5 , 6 , 10, 15, 30

$D_2(30) = \{ (1,2) , (1, 3), (1, 5), (1, 6), (1, 10) , (1, 15) , (1, 30) ,(2, 3) , (2, 5) , (2, 15) ,$

$(3, 5) , (3, 10) , (5, 6) \}$.

Order of $D_2(30) = 13. = O[D_2(p_1p_2p_3)]$ (A)

$D_3(30) = \{ (1,2, 3) ,(1 , 2 , 5) , (1, 3 , 5) , (2 , 3, 5) , (1, 3, 10) , (1 , 5, 6) , (1,2, 15) \}$

Order of $D_3(30) = 7$.

$D_4(30) = \{ (1,2, 3, 5)\}$, **Order of $D_4(30) = 1$.**

OPEN PROBLEM: To determine the order of $D_r(N)$.

In this note we consider the simple case of n being a **square-free number** for r = 2 , 3 etc.

(A) r = 2

We rather derive a reduction formula for r = 2. And finally a direct formula.

Let N = $p_1p_2p_3...p_n$ where p_k is a prime for k = 1 to n

We denote D_2 (N) = D_2 (1#n) for convnience. We shall derive a reductio formula for

D_2(1 # (n+1)).

Let q be a prime such that (q, N) = 1 , (HCF =1)

Then D_2(Nq) = D_2(1#(n+1))

1. We have by definition D_2(1#n) $\subset$ D_2(1#(n+1))

   This provides us with O [D_2(1#n)] elements of D_2(1#(n+1)).

   (2) Consider an arbitrarily chosen element (d_k , d_s) of D_2(1#n). This element when combined with q yields exactly two elements of D_2(1#(n+1)). i.e. (qd_k , d_s) and (d_k , qd_s).

   Hence the set D_2(1#n). contributes two times the order of itself.

2. The element (1, q) has ot been considered in the above mentioned cases hence the the total number of elements of D_2(1#(n+1)) are 3 times the order of D_2(1#n) + 1.

O[D_2(1#(n+1))] = 3 x O[D_2(1#n)] + 1.   (B)

Applying Reduction Formula (B) for evaluating O[D_2(1#4)]

From (A) we have O[D_2($p_1p_2p_3$)] = O[D_2(1#3)] = 13 hence

O[D_2(1#4)] = 3x13+ 1 = 40 .

This can be verified by considering N = 2x3x5x7 = 210. The divisors are

1 , 2 , 3 , 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210,

D_2(210) = { (1,2) , (1, 3) , (1, 5) , (1, 6) , (1, 7) , (1, 10), (1, 14) ,(1 ,15), (1,21),(1, 30),

(1, 35), (1, 42) , (1, 70) , (1, 105) , (1 , 210) , (2, 3) , (2, 5) , (2, 7) , (2, 15) , (2, 21) ,

(2, 35), (2, 105) , (3 , 5) , (3, 7) ,(3, 10), (3, 14) , (3, 35) , (3 , 70), (5 , 6), (5, 7) ,

(5, 14) , (5 , 21) , (5, 42) , (7, 6) , (7, 10) , (7, 15) ,(7, 30) , (6, 35) , (10, 21) , (14, 15) }

O [D_2(210)] = 40.

The reduction formula (B) can be reduced to a direct formula by applying simple induction and we get

O[D_2(1#n)] = $(3^n - 1)/2$ (C)

(B) r = 3.

For r = 3 we derive a reduction formula.

(1) We have $D_3(1\#n) \subset D_3(1\#(n+1))$ hence this contributes $O[D_3(1\#n)]$ elements to $D_3(1\#(n+1))$.

(2) Let us Choose an arbitrary element of $D_3(1\#n)$ say (a , b , c). The additional prime q yields (qa , b , c),

(a , qb , c), (a , b , qc) i.e. three elements. In this way we get 3 x $O[D_3(1\#n)]$ elements.

3. Let the product of the n primes = N . Let ($d_1, d_2, d_3, \ldots d_{d(N)}$) be all the divisors of N . Consider D_2 (1#n) which contains d(N) - 1 elements in which one member is unity = d_1. i.e. , (1, d_2) , (1, d_3) , . . ., (1, $d_{d(N)}$) .

If q is placed as the third element with these as the third element we get d(N) - 1 elements of D_3 (1#(n+1)). The remaining eleents of D_2 (1#n) yield elements repetitive elements already covered under (2).

Considering the exhaustive contributions from all the three above we get

$O[D_3(1\#(n+1))] = 4 * O[D_3(1\#n)] + d(N) - 1$

$\mathbf{O[D_3(1\#(n+1))] = 4 * O[D_3(1\#n)] + 2^n - 1}$ **(D)**

$O[D_3(210)] = 4 * O[D_3(30)]\ 8 - 1$

$\mathbf{O[D_3(210)] = 4 * 7 + 8 - 1 = 35}$

To verify the elements are listed below.

$\mathbf{D_3(210)}$ = { (1,2, 3) , (1, 2 , 5) , (1, 3, 5) , , (1, 2, 7) , (1, 3, 7), (1 ,5 , 7), (1, 2 , 15),(1, 2, 21),

(1, 2, 35), (1,2 , 105) , (1, 3, 10) , (1, 3, 14) , (1 , 3, 35) , (1, 3, 70) , (1, 5, 6) , (1, 5,14) , (1, 5 , 21) ,

(1, 5, 42), (1,7, 6) , (1 ,7, 10) , (1, 7, 15) ,(1, 7, 30), (2 ,3, 5) , (2, 3, 7) , (2 , 5, 7), (2 , 3 , 35), (2, 5, 21) ,

(2, 7 ,15) , (3, 5 , 7) , (3, 5, 14) , (3 , 7, 10) , (5, 7, 6) , (1, 6, 35) , (1, 10, 21) , (1, 14, 15) }

Open Problem : To obtain a direct formula from the reduction formula (D).

Regarding the general case i.e. O [D_r (1#n)] we derive an inequality.

Let ($d_1, d_2, d_3, \ldots d_r$) be an element of O [D_r (1#n)].

Introducing a new prime q other than the prime factors of N we see that this element in conjunction with q gives r elements of D_r (1#(n+1)) i.e. ($\mathbf{qd_1}$, d_2, d_3 , . . . d_r), (d_1, $\mathbf{qd_2}$, d_3 , . . . d_r), . . .

(d_1, d_2, d_3 , . . . $\mathbf{qd_r}$) .also D_r (1#n) $\subset$ D_r (1#(n+1)). Hence we get

O[D_r (1#(n+1))] > (r+1). O[D_r (1#n)]

To find an accurate formula is a tough task ahead for the readers.

Considering the general case is a further challenging job.