EXPANSION OF x^n IN SMARANDACHE TERMS OF PERMUTATIONS

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ABSTRACT:

DEFINITION of SMARANDACHE TERM

Consider the expansion of x^n as follows

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^xP_n \quad \text{---(9.1)}$$

We define $b_{(n,r)}\, x(x-1)(x-2)\ldots(x-r+1)(x-r)$ as the r^{th} **SMARANDACHE TERM** in the above expansion of x^n .

In the present note we study the coefficients $b_{(n,r)}$.of the the r^{th} **SMARANDACHE TERM** in such an expansion. We are encountered with fascinating coincidences.

DISCUSSION:

Let us examine the coefficients $b_{(n,r)}$.of the the r^{th} **SMARANDACHE TERM** in such an expansion.

Taking $x = 1$ gives $b_{(n,1)} = 1$

Taking $x = 2$ gives $b_{(n,2)} = (2^n - 2)/2$

Taking $x = 3$ gives $b_{(n,3)} = \{3^n - 3 - 6(2^n - 2)/2\}/6$

$$= \{1/3!\}\{ (1).3^n - (3).\ 2^n + (3).\ (1)^n - (1)\ (0)^n \}$$

Taking $x = 4$ gives

$$b_{(n,4)} = (1/4!)\,[\, (1)\, 4^n - (4)\, 3^n + (6)\, 2^n - (4)\, 1^n + 1(0)^n \,]$$

This suggests the possibility of

$$b_{(n,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} \cdot {}^{r}C_k \cdot k^n = a_{(n,r)}$$

THEOREM (9.1)

$$b_{(n,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} \cdot {}^{r}C_k \cdot k^n = a_{(n,r)}$$

First Proof:

This will be proved in two parts. First we shall prove the following proposition.

$$b_{(n+1,r)} = b_{(n,r-1)} + r.b_{(n,r)}$$

we have

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^{x}P_n$

$x = r$, gives,

$r^n = b_{(n,1)}\, r + b_{(n,2)}\, r(r-1) + b_{(n,3)}\, r(r-1)(r-2) + \ldots + b_{(n,n)}\, {}^{r}P_n$

multiplying both the sides by r ,

$r^{n+1} = b_{(n,1)}\, r.r + b_{(n,2)}\, r(r-1) + b_{(n,3)}\, r.r(r-1)(r-2) + \ldots + b_{(n,r)}\, r.\, {}^{r}P_r$ + terms equal to zero.

$r^{n+1} = b_{(n,1)}\, r.{}^{r}P_1 + b_{(n,2)}\, r.{}^{r}P_2 + b_{(n,3)}\, r.\, {}^{r}P_3 + \ldots + b_{(n,r)}\, r.\, {}^{r}P_r$

Using the identity $\mathbf{r.\, {}^{r}P_k = {}^{r}P_{k+1} + k.\, {}^{r}P_k}$ we can write

$r^{n+1} = b_{(n,1)} \{.{}^{r}P_2 + 1\,{}^{r}P_1\} + b_{(n,2)} \{{}^{r}P_3 + 2\, .{}^{r}P_2\} + \ldots + b_{(n,r)} \{ {}^{r}P_r + r.\, {}^{r}P_{r-1} \}$

$r^{n+1} = b_{(n,1)}\, {}^{r}P_1 + \{ b_{(n,1)} + 2.\, b_{(n,2)}\}\, {}^{r}P_2 + \{ b_{(n,2)} + 3.\, b_{(n,3)}\}\, {}^{r}P_3 + \ldots +$

$\{ b_{(n,r-1)} + r.\, b_{(n,r)}\}\, {}^rP_r$ ----------(9.2)

Otherwise also we have

$r^{n+1} = b_{(n+1,1)}\, {}^rP_1 + b_{(n+1,2)}\, .{}^rP_2 + b_{(n+1,3)}\, .\, {}^rP_3 + . . . + b_{(n+1,r)}\, .\, {}^rP_r$

The coefficients of ${}^rP_t$ ($t < r$) are independent of r hence these can seperately be equated giving us

$$\mathbf{b_{(n+1,r)} = b_{(n,r-1)} + r.b_{(n,r)}}$$

Now we shall proceed by induction. Let

$$b_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k}\, .{}^rC_k\, .k^n$$

$$b_{(n,r-1)} = (1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k}\, .{}^{r-1}C_k\, .k^n$$

be true. Then the sum $b_{(n,r-1)} + r.b_{(n,r)}$ equals

$$(1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k}\, .{}^{r-1}C_k\, .k^n + r.\, (1/r!) \sum_{k=0}^{r} (-1)^{r-k}\, .{}^rC_k\, .k^n$$

$$= ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1}(-1)^{-k}\, r\, \{ {}^{r-1}C_k - {}^rC_k\}k^n \right] + r^{n+1}/r!$$

$$= ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1}(-1)^{-k}\, \{ -k.\, {}^rC_k\}k^n \right] + r^{n+1}/r!$$

$$=(1/r!) \sum_{k=0}^{r-1} (-1)^{r-k}\, {}^rC_k\, k^{n+1}$$

which gives us

$$b_{(n+1,r)} = (1/r!) \sum_{k=0}^{r-1} (-1)^{r-k}\, {}^rC_k\, k^{n+1}$$

$b_{(n+1,r)}$ also takes the same form. Hence by induction the proof is complete.

Second Proof: This proof is totally based on a combinatorial approach . **This method also provides us with a proof of the Conecture (6.3) of ref. [3] as a by product.**

If n objects no two alike are to be distributed in x boxes, no two alike, the number of ways this can be done is $\mathbf{x}^n$ since there are k alternatives for disposals of the first object, k alternatives for the disposal of the second, and so on.

Alternately let us proceed with a different approach. Let us consider the number of distributions in which exactly a given set of r boxes is filled (rest are empty.). Let it be called f(n,r).

We derive a formula for f(n,r) by using the inclusion exclusion principle.The method is illustrated by the computation of f(n,5). Consider the total number of arrangements, 5^n of n different objects in 5 different boxes. Say that such an arrangement has property 'a'. In case the first box is empty, property 'b' incase the second box is empty, and similar property 'c', 'd', and 'e' . for the other three boxes respectively. To find the number of distributions with no box empty, we simply count the number of distributions having none of the properties 'a', 'b', 'c', . . .etc. We can apply the following formula.

$$\mathbf{N - {}^rC_1.N(a) + {}^rC_2.N(a,b) - {}^rC_3.N(a,b,c) +. . . ------(9.3)}$$

Here $N = 5^n$ is the total number of distributions. By N(a) we mean the number of distributions with the first box empty.and so $N(a) = 4^n$. Similarly N(a,b) is the number of distributions with the first two boxes empty. But this is the same as the number of distributions into 3 boxes and $N(a,b) = 3^n$. Thus we can write

$N = 5^n$, $N(a) = 4^n$, $N(a,b) = 3^n$ etc. $N(a,b,c,d,e) = 0$.

Applying formula (9.3) we get

$$f(n,5) = 5^n - {}^5C_1.4^n + {}^5C_2.3^n - {}^5C_3.2^n + {}^5C_4.1^n - {}^5C_5.0^n$$

by the direct generalization of this with r in place of 5 , we see that

$$\mathbf{f(n,r) = r^n - {}^rC_1.(r-1)^n + {}^rC_2.(r-2)^n - {}^rC_3.(r-3)^n + \ldots}$$

$$\mathbf{f(n,r) = \sum_{k=0}^{r} (-1)^k\, {}^rC_k\, (r-k)^n}$$

$\mathbf{f(n,r) = r!\ .\ a_{(n,r)}}$ **, from theorem (3.1). of ref. [1]**

Now these r boxes out of the given x boxes can be chosen in ${}^xC_r$ ways. Hence the total number of ways in which n distinct objects distributed in x distinct boxes occupying exactly r of them (with the rest x-r boxes empty) , defined as **d(n,r/x)** , is given by

$$\mathbf{d(n,r/x) = r!\ .\ a_{(n,r)}\ {}^xC_r}$$

$$\mathbf{d(n,r/x) = a_{(n,r)}.\ {}^xP_r}$$

Summing up all the cases for r =0 to r = x , the total number of ways in which n distinct objects can be distributed in x distinct boxes is given by

$$\sum_{r=0}^{x} d(n,r/x)) = \sum_{r=0}^{x} {}^xP_r\, a_{(n,r)} \qquad \text{-------(9.4)}$$

equating the two results obtained by two different approaches we get

$$x^n = \sum_{r=0}^{n} {}^xP_r\, a_{(n,r)}$$

REMARKS:

If n distinct objects are to be distributed in x distinct boxes with no box empty , then n < x is mandetory for a possible distribution.e.g. 5 objects can not be placed in 7 boxes with no empty boxes (a sort of converse of peigon hole principle) Hence we get the following result

$$\mathbf{f(n,r) = 0, \quad for\ n < k.}$$

$$f(n,r) = \sum_{k=0}^{r} (-1)^k\, {}^rC_k\, (r-k)^n = 0 \text{ if } n < r.$$

Further Generalisation:

(1) One can go ahead with the following generalisation of expansion of x^n as follows

$$x^n = g_{(n/k,1)}\, x + g_{(n/k,2)}\, x(x-k) + g_{(n/k,3)}\, x(x-k)(x-2k) + \ldots + g_{(n/k,n)}\, x(x-k)(x-2k)\ldots(x-(n-1)k)(x-nk+k)$$

$g_{(n/k,r)} = b_{(n,r)} = a_{(n,r)}$ for k = 1 has been dealt with in this note. One can explore for beautiful patterns for k = 2 , 3 etc. We can call (define) $g_{(n/k,r)}\, x(x-k)(x-2k)\ldots(x-(n-1)k)(x-rk+k)$ as the r^{th} **Smarandache Term of the** k^{th} **kind** in such an expansion.

(2) Another generalisation could be

$$x^{n!} = c_{(n/k,1)}(x-k) + c_{(n/k,2)}(x-k)(x^2-k) + c_{(n/k,3)}(x-k)(x^2-k)(x^3-k) +\ldots+\ldots+ c_{(n/k,n)}(x-k)(x^2-k)(x^3-k)\ldots(x^n-k)$$

For $k = 1$ if we denote $c_{(n/k,r)} = c_{(n,r)}$ for simplicity we get

$$x^{n!} = c_{(n,1)}(x-1) + c_{(n,2)}(x-1)(x^2-1) + c_{(n,3)}(x-1)(x^2-1)(x^3-1) +\ldots+\ldots+ c_{(n,n)}(x-1)(x^2-1)(x^3-1)\ldots(x^n-1)$$

We can define $c_{(n/k,r)}.(x-k)(x^2-k)(x^3-k)\ldots(x^r-k)$ as the **r^{th} Smarandache Factorial Term of the k^{th} kind** in the expansion of $x^{n!}$. One can again explore for patterns for the coefficient $c_{(n/k,r)}$.

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