Formula to obtain the next prime in an arithmetic progression

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Abstract: In this article, a formula is given to obtain the next prime in an arithmetic progression.

Theorem: We consider the arithmetic progression $a+di \quad i \geq 0$ of positive integers with $GCD(a,d)=1$ and considering that the final term is $a+dM$ is to say $0 \leq i \leq M$.
Let p a term in the arithmetic progression (it doesn't have to be prime).
Then the next prime in the arithmetic progression is:

$$nxt(a,d)(p) = p+d+d\cdot \sum_{k=1+(p-a)/d}^{M} \prod_{j=1+(p-a)/d}^{k} \left\lceil -\left\lfloor \frac{2-\sum_{s=1}^{a+jd}\left(\lfloor (a+jd)/s \rfloor - \lfloor (a+jd-1)/s \rfloor\right)}{a+jd} \right\rfloor \right\rceil$$

and the improved formula:

$$nxt(a,d)(p) = p+d+d\cdot \sum_{k=1+(p-a)/d}^{M} \prod_{j=1+(p-a)/d}^{k} \left[-\left(\left(2+2\sum_{s=1}^{\sqrt{a+jd}}\left((a+jd-1)/s-(a+jd)/s\right)\right)/(a+jd)\right)\right]$$

Where $\lfloor x \rfloor$ = is the floor function. And where x/y is the integer division in the improved formula.
Proof:

By a past article [1] we have that the next prime function is:

$$nxt(p) = p+1+\sum_{k=p+1}^{2p} \prod_{i=p+1}^{k} \left\lceil -\left\lfloor \frac{2-\sum_{s=1}^{i}\left(\left\lfloor \frac{i}{s} \right\rfloor - \left\lfloor \frac{i-1}{s} \right\rfloor\right)}{i} \right\rfloor \right\rceil$$

Where the expression of the product is the Smarandache Prime Function:

$$G(i) = \begin{cases} 1 & if \quad i \quad is\ composite \\ 0 & if \quad i \quad is\ prime \end{cases}$$

We consider that $a+j_0 d$ is the next prime of a number p in an arithmetic progression $a+jd$. We have that $G(a+j_0 d)=0$.
And for all j such that $p<a+jd<a+j_0 d$ we have that $G(a+jd)=1$.

It is deduced that:

$$\prod_{j=1+(p-a)/d}^{k} G(a+jd)=\prod_{j=1+(p-a)/d}^{j_0-1} G(a+jd)\cdot\prod_{j\geq j_0}^{k} G(a+jd)=\begin{cases}0 & k>j_0-1\\ 1 & k\leq j_0-1\end{cases}$$

since the first product has the value of 1, and the second product is zero since it has the factor $G(a+j_0 d)=0$.

As a result in the formula $nxt(a,d)$ the non zero terms are summed until j_0-1 and has the value of 1.

$$nxt(a,d)(p)=p+d+d\cdot\sum_{k=1+(p-a)/d}^{j_0-1} 1=p+d+d\cdot(j_0-1+1-1-(p-a)/d)=$$

$$=p+d+j_0 d-d-p+a=a+j_0 d$$

And the result is proven.

The improved formula [2] is obtained by considering that the sum, in the Smarandache prime function, until the integer part of the square root and multiplied by 2 the result. Also the floor function is changed $\lfloor x\rfloor$ for the integer division operator x/y that it faster for the computation.
Let us see an example made in MATHEMATICA:

```
a=5
5
dd=4
4
M=20
20
p=5
5
DD[i_]:=Sum[Quotient[(a+i*dd),j]-Quotient[a+i*dd-1,j],
{j,1,Sqrt[a+i*dd]}]
G[i_]:=-Quotient[(2-2*DD[i]),(a+i*dd)]
F[m_]:=Product[G[i],{i,(p-a)/dd+1,m}]
S[n_]:=Sum[F[m],{m,(p-a)/dd+1,M}]
While[p<a+(M-1)*dd+1,Print["nxt(",p,")=",p+dd+dd*S[p]];
p=p+dd+dd*S[p]]
nxt(5)=13
nxt(13)=17
nxt(17)=29
nxt(29)=37
nxt(37)=41
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nxt(41)=53
nxt(53)=61
nxt(61)=73
nxt(73)=89
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The question is that if these formulas can be applied to prove the Dirichlet's Theorem [3]for arithmetic progressions.

That is to say: does any arithmetic progression $a+jd$ such that $GCD(a,d)=1$ have infinite primes?

REFERENCES:

[1] ***S M Ruiz***. A functional recurrence to obtain the prime numbers using the Smarandache prime function. Smarandache Notions J., Vol. 11, No. 1-2-3, Spring 2000, p. 56.
http://personal.telefonica.terra.es/web/smruiz/

[2] ***Carlos Rivera***. The Prime Puzzles & Problems Connection. Problem 38. www.primepuzzles.net

[3] ***Thomas Koshy***. Elementary Number Theory with applications. Page 178.