Formula to Obtain the Next Prime Number in Any Increasing Sequence of Positive Integer Numbers

Sebastián Martín Ruiz
Avda. de Regla 43, Chipiona 11550 Spain
smaranda@teleline.es
www.telefonica.net/c/smruiz

Abstract:

In this article I give a generalization of the previous formulas [1],[2],[3] to obtain the following prime number, valid for any increasing sequence of positive integer numbers in the one that we know the algebraic expression of its nth term.

THEOREM: Let $\{a_n\}_{n\geq 1}$ an increasing sequence of positive integers of which we know the algebraic expression of its nth term; that is to say:

It exists $f: N \to N$ such that $f(n) = a_n$

As f it is increasing also:

It exists $f^{-1}: N \to R$ inverse function of f

Let $p \in \{a_n\}_{n\geq 1}$ a term of the sequence, (It doesn't have to be prime)

Let us consider the expression obtained by me [1],[2],[3],[4] of the Smarandache prime function:

$$G(k) = -\left\lfloor \frac{2 - \left(\sum_{s=1}^{k} \left\lfloor \frac{k}{s} \right\rfloor - \left\lfloor \frac{k-1}{s} \right\rfloor \right)}{k} \right\rfloor = \begin{cases} 1 & \text{if } k \text{ is composite} \\ 0 & \text{if } k \text{ is prime} \end{cases}$$

$\lfloor x \rfloor$ = is the greatest integer less than or equal to x.

And their improved expression [3]:

$$G(k) = -\left[\left(2 + 2\sum_{s=1}^{\sqrt{k}} ((k-1)/s - k/s) \right) / k \right]$$ where all the divisions of this last expression are integer divisions.

Then the next prime number in the sequence is:

$$NXT_f(p) = f\left[f^{-1}(p) + 1 + \sum_{k \geq f^{-1}(p)+1} \prod_{j=f^{-1}(p)+1}^{k} G(f(j)) \right]$$

REMARKS:

1: It is necessary that $p \in \{a_n\}$ so that $f^{-1}(p)$ it is an integer number.

2: Although the sum in the expression NXT_f is not enclosed, we can calculate the sum until a given M, to obtain a computable algorithm that then will see .(Examples). The only inconvenience is , that when making this truncation, the last value obtained in the algorithm is not correct in general , but all the other values are correct. (Examples).

3:With the improved expression of G(k) the calculation is much quicker.

4:The function is increasing in strict sense. i.e.:

$$j < k \Rightarrow a_j < a_k \text{ and } a_j \neq a_k$$

Many sequences of integers numbers are increasing in strict sense.

5: The algorithm that is obtained is of polynomial complexity if f is of polynomial complexity (Examples).

PROOF:

Let $p \in \{a_n\}_{n\geq 1} = \{f(n)\}_{n\geq 1}$ as we already said, It doesn't have to be prime. $p = f(j_0)$ with $j_0 \geq 1$.

$$NXT_f(p) = f\left[j_0 + 1 + \sum_{k \geq j_0+1} \prod_{j=j_0+1}^{k} G(f(j)) \right]$$

$$\sum_{k\geq j_0+1} \prod_{j=j_0+1}^{k} G(f(j)) = \sum_{k=j_0+1}^{j_1-1} \prod_{j=j_0+1}^{k} G(f(j)) + \sum_{k\geq j_1} \prod_{j=j_0+1}^{k} G(f(j)) = (**)$$

Where $f(j_1) = q$ is the next prime number to $f(j_0) = p$ in the sequence $\{f(n)\}_{n\geq 1}$.

$j_0 + 1 \leq j \leq j_1 - 1$ and $f(n)$ increasing, it implies that

$p < f(j) < q \quad j : j_0 + 1 \leq j \leq j_1 - 1$.

Therefore $f(j)$ is composite for all $j : j_0 + 1 \leq j \leq j_1 - 1$,for which:

$G(f(j)) = 1 \quad j : j_0 + 1 \leq j \leq j_1 - 1$.

On the other hand $G(f(j_1)) = 0 = G(q)$ since q is prime.

Returning to the previous expression has that:

$$(**) = j_1 - 1 - (j_0 + 1) + 1 + \sum_{k\geq j_1} 0 = j_1 - j_0 - 1$$

Lastly we have that:

$NXT_f(p) = f(j_0 + 1 + j_1 - j_0 - 1) = f(j_1) = q$ and the theorem is already proved.

EXAMPLES: I give three examples of the algorithm in MATHEMATICA language.:

Example 1:

```
M = 40
40
f[n_] := n^2 + 3
f1[p_] := Sqrt[p - 3]
G[x_] := -Quotient[(2 +
      2*Sum[Quotient[(x - 1), s] - Quotient[x, s], {s, 1, Sqrt[x]}]), x]
NXT[p_] :=
 f[f1[p] + 1 + Sum[Product[G[f[j]], {j, f1[p] + 1, k}], {k, f1[p] + 1, M}]]
p = f[1]
4
While[p < f[M], (Print[NXT[p], " ", PrimeQ[NXT[p]]]; p = NXT[p])]
7     True
19    True
67    True
103   True
199   True
487   True
787   True
1447  True
1684  False
```

It is observed that the last value is not correct due to the truncation.

Example 2:

```
M = 40
40
f[n_] := n^3 + 4
f1[p_] := (p - 4)^(1/3)
G[x_] := -Quotient[(2 +
      2*Sum[Quotient[(x - 1), s] - Quotient[x, s], {s, 1, Sqrt[x]}]), x]
NXT[p_] :=
 f[f1[p] + 1 + Sum[Product[G[f[j]], {j, f1[p] + 1, k}], {k, f1[p] + 1, M}]]
p = f[1]
5
While[p < f[M], (Print[NXT[p], " ", PrimeQ[NXT[p]]]; p = NXT[p])]
31     True
347    True
733    True
6863   True
15629  True
19687  True
68925  False
```

It happens the same thing with the last value.

Example 3:

```
M = 125
125
f[n_] := n^2 + 1
f1[p_] := Sqrt[p - 1]
G[x_] := -Quotient[(2 +
       2*Sum[Quotient[(x - 1), s] - Quotient[x, s], {s, 1, Sqrt[x]}]), x]
NXT[p_] :=
  f[f1[p] + 1 + Sum[Product[G[f[j]], {j, f1[p] + 1, k}], {k, f1[p] + 1, M}]]
p = f[1]
2
While[p < f[M], (Print[NXT[p], " ", PrimeQ[NXT[p]]]; p = NXT[p])]
5      True
17     True
37     True
101    True
197    True
257    True
401    True
577    True
677    True
1297   True
1601   True
2917   True
3137   True
4357   True
5477   True
7057   True
8101   True
8837   True
12101  True
13457  True
14401  True
15377  True
15877  True
```

Except at most the last one, all the values obtained by the algorithm are correct.

REFERENCES:

[1] Sebastián Martín Ruiz, *A functional recurrence to obtain the prime numbers using the Smarandache prime function.* Smarandache Notions Journal Vol.11 page 56 (2000)
[2] Sebastián Martín Ruiz, *Formula to obtain the next prime in an arithmetic progression.* *http://www.gallup.unm.edu/~Smarandache/SMRuiz-nextprime.pdf*
[3] Carlos Rivera The prime Puzzles & Problems Connection. Problem 38. www.primepuzzles.net
[4] E. Burton. *Smarandache Prime and Coprime functions.* www.gallup.unm.edu/~smarandache/primfnct.txt