GENERALIZATION OF PARTITION FUNCTION, INTRODUCING SMARANDACHE FACTOR PARTITION

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Partition function P(n) is defined as the number of ways that a positive integer can be expressed as the sum of positive integers. Two partitions are not considered to be different if they differ only in the order of their summands. A number of results concerning the partition function were discovered using analytic functions by Euler, Jacobi, Hardy , Ramanujan and others. Also a number of congruence properties of the function were derived. In the paper Ref.[1]

"SMARANDACHE RECIPROCAL PARTITION OF UNITY SETS AND SEQUENCES"

while dealing with the idea of Smarandache Reciprocal Partitions of unity we are confronted with the problem as to in how many ways a number can be expressed as the product of its divisors. Exploring this lead to the generalization of the theory of partitions.

DISCUSSION:

Definition : SMARANDACHE FACTOR PARTITION FUNCTION:

Let α_1 , α_2 , α_3 , . . . α_r be a set of r natural numbers and p_1 , p_2, p_3 ,. . .p_r be arbitrarily chosen distinct primes then

$F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1}\ p_2^{\alpha 2}\ p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors.

Example: $F(1,2) = ?$,

Let $p_1 = 2$ and $p_2 = 3$, $N = p_1 . p_2^2 = 2.3^2 = 18$

N can be expressed as the product of its divisors in following 4 ways:

(**1**) N = 18, (**2**) N = 9 X 2

(**3**) N = 6 X 3 (**4**) N = 3 X 3 X 2. As per our definition F (1,2) = 4.

It is evident from the definition that $F(\alpha_1, \alpha_2) = F(\alpha_2, \alpha_1)$ or in general the order of α_i in $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i \ldots \alpha_r)$ is immaterial. Also the primes $p_1, p_2, p_3, \ldots p_r$ are dummies and can be chosen arbitrarily.

We start with some elementry results to buildup the concept.

THEOREM(2.1) : F (α) = P(α)

where $P(\alpha)$ is the number of partitions of α.

PROOF: Let p be any prime and $N = p^{\alpha}$.

Let $\alpha = x_1 + x_2 + \ldots + x_m$ be a partition of α.

Then $N = (p^{x1})(p^{x2})(p^{x3}) \ldots (p^{xn})$ is a **SFP** of N .i.e. each partition of α contributes one **SFP.** ---------(2.1)

Also let one of the **SFP** of N be

$N = (N_1).(N_2.)(N_3) \ldots (N_k)$. Each N_i has to be such that $N_i = p^{ai}$

Let $N_1 = p^{a1}$, $N_2 = p^{a2}$, etc. $N_k = p^{ak}$ then

$$N = (p^{a1})(p^{a2}) \ldots (p^{ak})$$

$$N = p^{(a1+a2+a3+\ldots+an)}$$

$$\Rightarrow \quad \alpha = a_1 + a_2 + \ldots + a_k$$

which gives a partition of α. Obviously each **SFP** of N gives one unique partition of α . --------(2.2) .

from (2.1) and (2.2) we get

$$\mathbf{F(\alpha) = P(\alpha)}$$

THEOREM (2.2) $\quad \mathbf{F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)}$

PROOF: Let $N = p_1^{\alpha} p_2$, where p_1, p_2 are arbitrarily chosen primes.

<u>Case(1)</u> Writing $N = (p_2)\, p_1^{\alpha}$ keeping p_2 as a separate entity (one of the factors in the factor partition of N) ,would yield $P(\alpha)$ Smarandache factor partitions .(from theorem (2.1)) .

<u>Case(2)</u> Writing $N = (p_1.p_2).\, p_1^{\alpha-1}$ keeping (p_1p2) as a separate entity (one of the factors in the SFP of N) ,would yield $P(\alpha-1)$ SFPs.

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Case (r) In general writing $N = (p_1^r . p_2). p_1^{\alpha-r}$ and keeping $(p_1^r . p_2)$ as a separate entity would yield $P(\alpha-r)$ SFPs.

Contributions towards F(N) in each case (1), (2), . . . (r) are mutually disjoint as $p_1^r . p_2$ is unique for a given r . which ranges from zero to α. These are exhaustive also.

Hence

$$F(\alpha, 1) = \sum_{r=0}^{\alpha} P(\alpha-r)$$

Let $\alpha - r = k$ $\quad r = 0 \Rightarrow k = \alpha$

$r = \alpha \Rightarrow k = 0$

$$F(\alpha, 1) = \sum_{k=\alpha}^{0} P(k)$$

$$F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)$$

This completes the proof of the theorem (2.2)

Some examples:

(1) F(3) = P(3) = 3 , Let p = 2 , $N = 2^3 = 8$

(I) N = 8 , (2) N = 4 X 2 , (3) N = 2 X 2 X 2 .

(2)

$$F(4, 1) = \sum_{k=0}^{4} P(k) = P(0) + P(1) + P(2) + P(3) + P(4)$$

$$= 1 + 1 + 2 + 3 + 5 = 12$$

Let $N = 2^4 \times 3 = 48$ here $p_1 = 2$, $p_2 = 3$.

The Smarandache factor partitions of 48 are

(1) N = 48
(2) N = 24 X 2
(3) N = 16 X 3
(4) N = 12 X 4
(5) N = 12 X 2 X 2
(6) N = 8 X 6
(7) N = 8 X 3 X 2
(8) N = 6 X 4 X 2
(9) N = 6 X 2 X 2 X 2
(10) N = 4 X 4 X 3
(11) N = 4 X 3 X 2 X 2
(12) N = 3 X 2 X 2 X 2 X 2

DEFINITIONS:
In what follows in the coming pages let us denote (for simplicity)

(1) $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$

where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

(2) Also for the case (N is a square free number)

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

Let us denote

$$F(1, 1, 1, 1, 1 \ldots) = F(1\#n)$$
$\leftarrow$ n - ones $\rightarrow$

Examples: F (1#2) = F(1, 1) = F'(6) = 2 , $6 = 2 \times 3 = p_1 \times p_2$.

F (1#3) = F (1 , 1, 1) = F' (2 X 3 X 5) = F' (30) = 5.

(3) **Smarandache Star Function**

$$F^{'*}(N) = \sum_{d/N} F'(d_r) \qquad \text{where } d_r | N$$

$F^{'*}(N)$ = sum of $F'(d_r)$ over all the divisors of N.

e.g. N = 12 , divisors are 1, 2, 3, 4, 6, 12

$F^{'*}(12) = F'(1) + F'(2) + F'(3) + F'(4) + F'(6) + F'(12)$

$= 1 + 1 + 1 + 2 + 2 + 4 = 11$

THEOREM (2.3)

$$F^{'*}(N) = F'(Np) \quad , \quad (p,N) = 1, \text{ p is a prime.}$$

PROOF: We have by definition

$$F^{'*}(N) = \sum_{d/N} F'(d_r) \qquad \text{where } d_r | N$$

consider d_r a divisor of N . $Np = d_r\,(Np/d_r)$

let $(Np/d_r) = g(d_r)$, then $N = d_r.\, g(d_r)$

for any divisor d_r of N , $g(d_r)$ is unique

$d_i = d_j \Leftrightarrow g(d_i) = g(d_j)$

Considering $g(d_r)$ as a single term (an entity , not further split into factors) in the SFP of **N.p** one gets $F'(d_r)$ SFPs .

Each $g(d_r)$ contributes $F'(d_r)$ factor partitions .

The condition p does not divide N , takes care that $g(d_i) \neq d_j$ for any divisor. because p divides $g(d_i)$ and p does not divide d_j.

This ensures that contribution towards $F'(Np)$ from each $g(d_r)$ is distinct and there is no repetition. Summing over all $g(d_r)$'s we get

$$F'(Np) = \sum_{d/N} F'(d_r)$$

or

$$F'^{*}(N) = F'(Np)$$

This completes the proof of the theorem (3) .

An application of theorem (2.3)

Theorem (2.2) follows from theorem (2.3)
To prove

$$F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)$$

Let $N = p^{\alpha} p_1$ then $F(\alpha,1) = F'(p^{\alpha} . p_1)$

from theorem (2.3)

$$F'(p^{\alpha} . p_1) = F'^{*}(p^{\alpha}) = \sum_{d/p^{\alpha}} F'(d_r)$$

The divisor of p^{α} are $p^0, p^1, p^2, \ldots p^{\alpha}$

hence

$$F'(p^{\alpha} . p_1) = F'(p^0) + F'(p^1) + \ldots + F'(p^{\alpha})$$

$$= P(0) + P(1) + P(2) + \ldots + P(\alpha - 1) + P(\alpha)$$

or

$$F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)$$

THEOREM (2.4):

$$F(1\#(n+1)) = \sum_{r=0}^{n} {}^{n}C_r F(1\#r)$$

PROOF: From theorem (2.3) we have $F'(Np) = F'^{*}(N)$, p does not divide N. Consider the case $N = p_1 p_2 p_3 \ldots p_n$. We have , $F'(N)$

= F (1#n) and F'(Np) = F (1#(n+1)) as p does not divide N.

Finally we get

$$F\ (1\#(n+1)) = F'^{*}\ (N) \qquad \text{--------(2.3)}$$

The number of divisors of N of the type $p_1p_2p_3 \ldots p_r$. (containing exactly r primes is ${}^nC_r$. Each of the ${}^nC_r$ divisors of the type $p_1p_2p_3 \ldots p_r$ has the same number of SFPs F(1#r). Hence

$$F'^{*}\ (N) = \sum_{r=0}^{n} {}^nC_r\ F(1\#r) \qquad \text{-------------(2.4)}$$

From (2.3) and (2.4) we get

$$F\ (\ 1\#\ (n+1)) = \sum_{r=0}^{n} {}^nC_r\ F(1\#r)$$

NOTE: It is to be noted that F(1#n) is the n^{th} Bell number.

Example: F(1#0) =F'(1) = 1.

F(1#1) = F'(p_1) = 1.

F(1#2) = F'(p_1 p_2) = 2.

F(1#2) = F'(p_1 p_2 p_3) = 5.

(i) $p_1\ p_2\ p_3$
(ii) $(p_1\ p_2) \times p_3$
(iii) $(p_1p_3) \times p_2$
(iv) $(p_2p_3) \times p_1$
(v) $p_1 \times p_2 \times p_3$

Let Theorem (4) be applied to obtain F (1#4)

$$F(1\#4) = \sum^{3} {}^nC_r\ F(1\#r)$$

r=0

$F(1\#4) = {}^3C_0\,F(1\#0) + {}^3C_1 F(1\#1) + {}^3C_2\,F(1\#2) + {}^3C_3\,F(1\#3)$

$= 1 X 1 + 3 X 1 + 3 X 2 + 1 X 5 = 15$

$F(1\#4) = F'(2 X 3 X 5 X 7) = F'(210) = 15.$

(I) 210
(ii) 105 X 2
(iii) 70 X 3
(iv) 42 X 5
(v) 35 X 6
(vi) 35 X 3 X 2
(vii) 30 X 7
(viii) 21 X 10
(ix) 21 X 5 X 2
(x) 15 X 14
(xi) 15 X 7 X 2
(xii) 14 X 5 X 3
(xiii) 10 X 7 X 3
(ixv) 7 X 6 X 5
(xv) 7 X 5 X 3 X2

On similar lines one can obtain

F(1#5) = 52 , F(1#6) = 203 , F(1#7) = 877 , F(1#8) = 4140.

F(1#9) = 21,147.

DEFINITION:

$$\mathbf{F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)}$$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors , let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Example:

$$F'^{**}(p_1p_2p_3) = F^{**}(1\#3) = \sum_{d_r/N} F'(d_r)$$

$$= {}^3C_0\, F'^*(1) + {}^3C_1 F'^*(p_1) + {}^3C_2\, F'^*(p_1p_2) + {}^3C_3\, F'^*\,(p_1p_2p_3)$$

$$F'^*(1\#3) = 1 + [\,3F'(1) + F'(p_1)] + 3[\,F'(1) + 2F'(p_1) + F'(p_1p_2)]$$

$$+ [\,F'(1) + 3F'(p_1) + 3F'(p_1p_2) + F'\,(p_1p_2p_3)\,]$$

$$F'^*(1\#3) = 1 + 6 + 15 + 15 = 37.$$

An interesting observation is

(1) $F^{**}(1\#0) + F(1\#1) = F(1\#2)$
or
$F^{**}(1\#0) + F^*(1\#0) = F(1\#2)$

(2) $F^{**}(1\#1) + F(1\#2) = F(1\#3)$
or
$F^{**}(1\#1) + F^*(1\#1) = F(1\#3)$

(3) $F^{**}(1\#5) + F(1\#6) = F(1\#7)$
or
$F^{**}(1\#5) + F^*(1\#5) = F(1\#7)$

which suggests the possibility of

$$\mathbf{F^{**}(1\#n) + F^*(1\#n) = F(1\#(n+2))}$$

A stronger proposition

$$F'(Np_1p_2) = F'^*(N) + F'^{**}(N)$$

is established in theorem (2.5).

DEFINITION:

$$F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r) \qquad n > 1$$

where $F'^*(N) = \sum_{d_r/N} F'(d_r)$

and d_r ranges over all the divisors of N.

THEOREM(2.5) :

$$F'(Np_1p_2) = F'^*(N) + F'^{**}(N)$$

from theorem (3) we have

$F'(Np_1p_2) = F'^*(Np_1)$

Let $d_1, d_2, \ldots, d_n$ be all the divisors of N. The divisors of Np_1 would be

$d_1, \quad d_2, \quad \ldots, \quad d_n$

$d_1p_1, d_2p_1, \ldots, d_np_1$

$$F'^*(Np_1) = [F'(d_1) + F'(d_2) + \ldots + F'(d_n)] + [F'(d_1p_1) + F'(d_2p_1) + \ldots + F'(d_np_1)]$$

$$= F'^*(N) + [F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n)]$$

$$F'^*(Np_1) = F'^*(N) + F'^{**}(N) \qquad \text{(by definition)}$$

$$= F'^*(N) + F'^{2*}(N)$$

This completes the proof of theorem (2.5).

THEOREM(2.6):

$$\mathbf{F'(Np_1p_2p_3) = F'^*(N) + 3F'^{2*}(N) + F'^{3*}(N)}$$

PROOF:

From theorem (2.3) we have

$F'(Np_1p_2p_3) = F'^*(Np_1p_2)$.

Also If $d_1, \quad d_2, \quad \ldots, \quad d_n$ be all the divisors of N. Then the divisors of Np_1p_2 would be

$$d_1, \quad d_2, \quad \ldots, \quad d_n$$

$$d_1p_1 \quad d_2p_1, \quad \ldots, \quad d_np_1$$

$$d_1p_2, \quad d_2p_2, \quad \ldots, \quad d_np_2$$

$$d_1p_1p_2, \quad d_2p_1p_2, \quad \ldots, \quad d_np_1p_2$$

Hence

$$F'^*(Np_1p_2) = [F'(d_1) + F'(d_2) + \ldots + F'(d_n)] + [F'(d_1p_1) + F'(d_2p_1) + \ldots F'(d_np_1)] + [F'(d_1p_2) + F'(d_2p_2) + \ldots + F'(d_np_2)] + [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)]$$

$$= F'^*(N) + 2[F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n)] + S \qquad \text{----(2.5)}$$

where $S = [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)$

Now from theorem (2.5) we get,

$$F'(d_1p1p2) = F'^*(d_1) + F'^{**}(d_1) \qquad \text{----(1)}$$

$$F'(d_2p_1p_2) = F'^*(d_2) + F'^{**}(d_2) \qquad \text{----(2)}$$

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$$F'(d_np1p2) = F'^*(d_n) + F'^{**}(d_n) \qquad \text{----(n)}$$

on summing up (1) , (2) . . . upto (n) we get

$$S = F'^{2*}(N) + F'^{3*}(N) \qquad \text{----(2.6)}$$

substituting the value of S in (A) and also taking

$F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n) = F'^{2*}(N)$

we get.,

$$F'(Np_1p_2p_3) = F'^{*}(N) + 2F'^{2*}(N) + F'^{2*}(N) + F'^{3*}(N)$$

$$F'(Np_1p_2p_3) = F'^{*}(N) + 3F'^{2*}(N) + F'^{3*}(N)$$

This completes the proof of theorem (2.6).The above result which has been observed to follow a beautiful pattern can further be generalized.

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