MISCELLANEOUS RESULTS AND THEOREMS ON SMARANDACHE TERMS AND FACTOR PARTITIONS

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ABSTRACT: In **[1]** we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let $\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r$ be a set of r natural numbers and $p_1 , p_2, p_3 ,\ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1}\; p_2^{\alpha 2}\; p_3^{\alpha 3} \ldots p_r^{\alpha r}$$

could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1}\; p_2^{\alpha_2}\; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

we denote

$$\underbrace{F (1 , 1, 1, 1, 1 \ldots)}_{\leftarrow \text{ n - ones } \rightarrow} = F (1\#n)$$

In **[2]** we define $\mathbf{b_{(n,r)}\, x(x-1)(x-2)\ldots(x-r+1)(x-r)}$ as the $\mathbf{r^{th}}$ **SMARANDACHE TERM** in the expansion of

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^xP_n$$

In this note some more results depicting how closely the coefficients of the **SMARANDACHE TERM** and **SFPs** are related. are derived.

DISCUSSION:

Result on the $[i^j]$ matrix:

Theorem (9.1) in **[2]** gives us the following result

$$x^n = \sum_{r=0}^{n} {}^xP_r\, a_{(n,r)}$$ which leads us to the following

beautiful result.

$$\sum_{k=1}^{x} k^n = \sum_{k=1}^{x} \sum_{r=1}^{k} {}^kP_r\, a_{(n,r)}$$

In matrix notation the same can be written as follows for $x = 4 = n$.

$$\begin{bmatrix} {}^1P_1 & 0 & 0 & 0 \\ {}^2P_1 & {}^2P_2 & 0 & 0 \\ {}^3P_1 & {}^3P_2 & {}^3P_3 & 0 \\ {}^4P_1 & {}^4P_2 & {}^4P_3 & {}^4P_4 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1^1 & 1^2 & 1^3 & 1^4 \\ 2^1 & 2^2 & 2^3 & 2^4 \\ 3^1 & 3^2 & 3^3 & 3^4 \\ 4^1 & 4^2 & 4^3 & 4^4 \end{bmatrix}$$

In gerneral

$$\mathbf{P} * \mathbf{A'} = \mathbf{Q} \quad \text{where } \mathbf{P} = \left[{}^iP_j\right]_{nXn} \quad \text{-----(10.1)}$$

$$A = \left[a_{(i,j)}\right]_{nXn} \quad \text{and} \quad \mathbf{Q} = \left[i^j\right]_{nXn}$$

(**A'** is the transpose of **A**)

Consider the expansion of x^n, again

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^{x}P_n$

for $x = 3$ we get

$x^3 = b_{(3,1)}\, x + b_{(3,2)}\, x(x-1) + b_{(3,3)}\, x(x-1)(x-2)$

comparing the coefficient of powers of x on both sides we get

$$b_{(3,1)} - b_{(3,2)} + 2\, b_{(3,3)} = 0$$

$$b_{(3,2)} - 3\, b_{(3,3)} = 0$$

$$b_{(3,3)} = 1$$

In matrix form

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} b_{(3,1)} \\ b_{(3,2)} \\ b_{(3,3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_3 * A_3 = B_3$$

$$A_3 = C_3^{-1} . B_3$$

$$C_3^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(C_3^{-1})' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

similarly it has been observed that

$$(C_4^{-1})' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}$$

The above observation leads to the following theorem.

THEOREM (10.1)

In the expansion of x^n as

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^xP_n$

If C_n be the coefficient matrix of equations obtained by equating the coefficient of powers of x on both sides then

$$(C_n^{1})' = \left[a_{(i,j)}\right]_{nXn} = \text{star matrix of order n}$$

PROOF: It is evident that C_{pq} the element of the p^{th} row and q^{th} column of C_n is the coefficient of x^p in ${}^xP_q$. And also C_{pq} is independent of n. The coefficient of x^p on the RHS is

coefficient of $x^p = \sum_{q=1}^{n} b_{(n,q)}\, C_{pq}$, also

coefficient of x^p = 1 if p = n

coefficient of x^p = =0 if $p \neq n$.

in matrix notation

$$\text{coefficient of } x^p = \left[\sum_{q=1}^{n} b_{(n,q)} C_{pq}\right]$$

$$= \left[\sum_{q=1}^{n} b_{(n,q)} C'_{qp}\right]$$

$= \; i_{np}$ where $i_{np} = 1$, if $n = p$ and $i_{np} = 0$, if $n \neq p$.

$= I_n$ (identity matrix of order n.)

$$\left[b_{(n,q)}\right]\left[C_{p,q}\right]' = I_n$$

$$\left[a_{(n,q)}\right]\left[C_{p,q}\right]' = I_n \qquad \text{as } b_{(n,q)} = a_{(n,q)}$$

$$A_n \,.\, C_n' = I_n$$

$$A_n \,.= I_n \,[\, C_n']^{-1}$$

$$A_n \,.= [\, C_n']^{-1}$$

This completes the proof of theorem (10.1).

THEOREM (10.2)

If $C_{k,n}$ is the coefficient of x^k in the expansion of ${}^xP_n$, then

$$\sum_{k=1}^{n} \mathbf{F(1\#k)\ C_{k,n} = 1}$$

PROOF: In property (3) of the STAR TRIANGLE following proposition has been established.

$F'(1\#n) = \sum_{m=1}^{n} a_{(n,m)} = B_n$, in matrix notation the same can be expressed as follows for $n = 4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix}$$

In general

$$\begin{bmatrix} 1 \end{bmatrix}_{1 \times n} * \underset{(c_n^{-1})'}{\begin{bmatrix} a_{(i,j)} \end{bmatrix}_{n \times n}} = \begin{bmatrix} B_i \end{bmatrix}_{1 \times n}$$

$$\begin{bmatrix} B_i \end{bmatrix}_{1 \times n} * \underset{(c_n)}{\begin{bmatrix} \quad \end{bmatrix}_{n \times n}} = \begin{bmatrix} 1 \end{bmatrix}_{1 \times n}$$

In $C_{n,n}$, $C_{p,q}$ the p^{th} row and q^{th} column is the coefficient of x^p in ${}^xP_q$. Hence we have

$$\sum_{k=1}^{n} F(1\#k)\, C_{k,n} = 1 = \sum_{k=1}^{n} B_k\, C_{k,n}$$

THEOREM(10.3)

$$\sum_{k=1}^{n} F(1\#(k+1))\, C_{k,n} = n + 1 = \sum_{k=1}^{n} B_{k+1}\, C_{k,n}$$

PROOF:
It has already been established that

$$B_{n+1} = \sum_{m=1}^{n} (m+1)\; a_{(n,m)}$$

In matrix notation

$$\begin{bmatrix} j+1 \end{bmatrix}_{1 \times n} * \underset{(c_n^{-1})}{\begin{bmatrix} a_{(i,j)} \end{bmatrix}_{n \times n}^{'}} = \begin{bmatrix} B_{j+1} \end{bmatrix}_{1 \times n}$$

$$\begin{bmatrix} j+1 \end{bmatrix}_{1 \times n} = \begin{bmatrix} B_{j+1} \end{bmatrix}_{1 \times n} * \underset{(c_n)}{\begin{bmatrix} \quad \end{bmatrix}_{n \times n}}$$

$$\sum_{k=1}^{n} B_{k+1}\, C_{k,n} = n + 1$$

There exist ample scope for more such results.

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