Near Pseudo Smarandache Function

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Abstract.

The *Pseudo Smarandache Functions* $Z(n)$ are defined by David Gorski [1].

This new paper defines a new *function K(n)* where $n \in N$, which is a slight modification of *Z(n)* by adding a *smallest natural number k*. Hence this function is *"Near Pseudo Smarandache Function (NPSF)"*.

Some properties of *K(n)* are presented here, separately, according to as *n* is *even* or *odd*. A *continued fraction* consisting *NPSF* is shown to be *convergent* [3]. Finally some properties of $K^{-1}(n)$ are also obtained.

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1.1 Definition

Near Pseudo Smarandache Function (*NPSF*) ***K*** is defined as follows.

$K : N \to N$ defined by $K(n) = m$, where $m = \Sigma n + k$ and k is the ***smallest natural number*** such that ***n divides m***.

1.2 Following are values of $K(n)$ for $n \leq 15$

n	Σn	k	$K(n)$
1	1	1	2
2	3	1	4
3	6	3	9
4	10	2	12
5	15	5	20
6	21	3	24
7	28	7	35
8	36	4	40
9	45	9	54
10	55	5	60
11	66	11	77
12	78	6	84
13	91	13	104
14	105	7	112
15	120	15	135

For more such values see appendix A

2.1 Properties

(i) $k = n$ **if** n **is** *odd* **and** $n/2$ **if** n **is** *even*.

(a) Let n be *odd*.

Then $(n + 1)$ is *even* and hence $(n + 1)/2$ is an *integer.*

$\therefore\ \Sigma n = n(n + 1)/2$, being multiple of n, is divisible by n.

Hence n divides $\Sigma n + k$ *iff n divides k* i.e. *iff* k is a multiple of n. However, as k is smallest $k = n$.

(b) Let n be *even*.

Then $\Sigma n + k = n(n+1)/2 + k = n^2/2 + n/2 + k$

As n is *even* hence $n/2$ is an *integer* and $n^2/2$ is *divisible* by n.

Hence *n divides* $\Sigma n + k$ *iff n divides* $n/2 + k$

i.e. iff $n \leq n/2 + k$ or $k \geq n/2$.

However, as k is smallest $k = n/2$.

(ii) **$K(n) = n(n+3)/2$ if n is *odd* and $K(n) = n(n+2)/2$ if n is *even*.**

$K(n) = \Sigma n + k = n(n+1)/2 + k$

If n is odd then $k = n$ and hence $K(n) = n(n+3)/2$

If n is *even* then $k = n/2$ and hence $K(n) = n(n+2)/2$.

(iii) **For all $n \in N$; $n(n+2)/2 \leq K(n) \leq n(n+3)/2$**

We know $K(n)$ is either $n(n+2)/2$ *Or* $n(n+3)/2$ depending upon whether n is *even* or *odd*.

Hence for all $n \in N$; $n(n+2)/2 \leq K(n) \leq n(n+3)/2$

(iv) **For all $n \in N$; $K(n) > n$.**

As $K(n) \geq n(n+2)/2 = n + n^2/2 > n$

Hence $K(n) > n$ for all $n \in N$.

(v) ***K* is *strictly monotonic increasing function* of *n*.**

Let m < n $\therefore m + 1 \leq n$ i.e. $m + (3-2) \leq n$

Or $m + 3 \leq n + 2$. So m < n and $m + 3 \leq n + 2$

$\therefore \quad m(m+3) < n(n+2)$

Or $m(m+3)/2 < n(n+2)/2$

$\therefore K(m) < K(n)$

Hence $K(n)$ is *strictly monotonic increasing function* of *n*.

(vi) **$K(m+n) \neq K(m) + K(n)$**

and **$K(m.n) \neq K(m).K(n)$**

We know $K(2) = 4$, $K(3) = 9$, $K(5) = 20$, & $K(6) = 24$

So $K(2) + K(3) = 4 + 9 = 13$ & $K(2+3) = K(5) = 20$

Hence $K(2+3) \neq K(2) + K(3)$

Also $K(2).K(3) = 4.9 = 36$ & $K(2.3) = K(6) = 24$

Hence $K(2.3) \neq K(2).K(3)$

2.2 (i) $\boldsymbol{K(2n+1) - K(2n) = 3n+2}$

$K(2n+1) = (2n+1)(2n+4)/2 = 2n^2+5n+2$

$K(2n) = 2n(2n+2)/2 = 2n^2+2n$

Hence $K(2n+1) - K(2n) = 3n+2$

(ii) $\boldsymbol{K(2n) - K(2m) = 2(n-m)(n+m+1)}$

$K(2n) = 2n(2n+2)/2 = 2n^2+2n$

$\therefore K(2n) - K(2m) = 2(\mathrm{n}^2-\mathrm{m}^2)+2(\mathrm{n}-\mathrm{m})$

Hence $K(2n) - K(2m) = 2(n-m)(n+m+1)$

(iii) $\boldsymbol{K(2n+1) - K(2n-1) = 4n+3}$

$K(2n+1) = (2n+1)(2n+4)/2 = 2n^2+5n+2$

$K(2n-1) = (2n-1)(2n+2)/2 = 2n^2+n-1$

Hence $K(2n+1) - K(2n-1) = 4n+3$

(iv) $\boldsymbol{K(n) - K(m) = \dfrac{n-m}{n+m} K(n+m)}$ ***where m, n are even and $n > m$.***

$$K(n)-K(m) = \frac{n}{2}(n+2) - \frac{m}{2}(m+2)$$

$$= \frac{1}{2}(n^2+2n-m^2-2m)$$

$$= \frac{1}{2}\{(n^2-m^2)+2(n-m)\}$$

$$= \left(\frac{n-m}{2}\right)(n+m+2)$$

$$= (n-m)\ \frac{1}{n+m}\ \frac{n+m}{2}\ (n+m+2)$$

$$= \frac{n-m}{n+m} K(n+m)$$

(v) **Let $K(n) = m$ and**

(a) **Let n be *even* then $n.m$ is a *perfect square iff* $(n+2)/2$ is a *perfect square.***

(b) **Let n be *odd* then $n.m$ is a *perfect square iff* $(n+3)/2$ is a *perfect square.***

(c) **$n.m$ is a *perfect cube iff* $n = 2$ *or* 3.**

(a) If n is even then $K(n) = m = n(n+2)/2$

$\therefore n.m = n^2(n+2)/2$ Hence if n is *even* then $n.m$ is a *perfect square iff* $(n+2)/2$ is a *perfect square.*

(b) If n is odd then $K(n) = m = n(n+3)/2$

$\therefore n.m = n^2(n+3)/2$ Hence if n is *odd* then $n.m$ is a *perfect square iff* $(n+3)/2$ is a *perfect square.*

(c) **Let n be *even* and let $n = 2p$**

Then $m = K(n) = K(2p) = 2p/2(2p+2)$

$\therefore n.m = (2p).p.2(p+1) = (2p).(2p).(p+1)$

$\therefore n.m$ *is a perfect cube iff* $p+1 = 2p$

i.e. iff $p = 1$ *i.e. iff* $n = 2$

Let n be *odd* and let $n = 2p - 1$

Then $m = K(n) = K(2p-1) = (2p-1)(2p-1+3)/2$

$= (2p-1)(p+1)$

$\therefore n.m = (2p-1).(2p-1).(p+1)$

$\therefore n.m$ *is a perfect cube iff* $p+1 = 2p-1$

i.e. iff $p = 2$ *i.e. iff* $n = 3$

$\therefore n = 2$ and $n = 3$ are the only two cases where $n.m$ is a *perfect cube.*

Verification :- $K(2) = 4$ & $2.4 = 8 = 2^3$

$K(3) = 9$ & $3.9 = 27 = 3^3$

2.3 Ratios

(i) $$\frac{K(n)}{K(n+1)} = \frac{n}{n+1}$$ **if n is odd.**

As n is odd $\therefore n+1$ is even. Hence $K(n) = n(n+3)/2$

and $K(n+1) = (n+1)(n+1+2)/2$

$= (n+1)(n+3)/2$

Hence $$\frac{K(n)}{K(n+1)} = \frac{n}{n+1}$$ if n is *odd.*

(ii) $$\frac{K(n)}{K(n+1)} = \frac{n(n+2)}{(n+1)(n+4)}$$ **if n is even.**

As n is even $\therefore n+1$ is odd . Also $K(n) = n(n+2)/2$ and

$K(n+1) = (n+1)(n+1+3)/2 = (n+1)(n+4)/2$

Hence $$\frac{K(n)}{K(n+1)} = \frac{n(n+2)}{(n+1)(n+4)}$$ if n is *even.*

(iii) $$\frac{K(2n)}{K(2n+2)} = \frac{n}{n+2}$$

$K(2n) = 2n(2n+2)/2 = 2n(n+1)$

$K(2n+2) = (2n+2)(2n+4)/2 = 2(n+1)(n+2)$

Hence $$\frac{K(2n)}{K(2n+2)} = \frac{n}{n+2}$$

2.4 Equations

(i) ***Equation*** $\boldsymbol{K(n) = n}$ **has no solution.**

We know $K(n) = n(n+2)/2$ **OR** $n(n+3)/2$

$\therefore K(n) = n$ *iff* $n(n+2)/2 = n$ **OR** $n(n+3)/2 = n$

i.e. *iff* $n = 0$ **OR** $n = -1$ which is not possible as $n \in \mathbb{N}$.

Hence *Equation* $K(n) = n$ has no solution.

(ii) ***Equation*** $\boldsymbol{K(n) = K(n+1)}$ **has no solution.**

If n is *even (or odd)* then $n+1$ is *odd (or even)*

Hence $K(n) = K(n+1)$

iff $n(n+2)/2 = (n+1)(n+4)/2$

OR $n(n+3)/2 = (n+1)(n+3)/2$

i.e. iff $n(n+2) = (n+1)(n+4)$

OR $n(n+3) = (n+1)(n+3)$

i.e. iff $n^2 + 2n = n^2 + 5n + 4$ **OR** $n^2 + 3n = n^2 + 4n + 3$

i.e iff $3n + 4 = 0$ **OR** $n + 3 = 0$

i.e iff $n = -4/3$ **OR** $n = -3$ which is not possible as $n \in N$.

Hence *Equation* $K(n) = K(n+1)$ has no solution.

(iii) ***Equation*** $K(n) = K(n+2)$ **has no solution.**

If n is *even* *(or odd)* then $n+2$ is *even* *(or odd)* .

Hence $K(n) = K(n+2)$

iff $n(n+2)/2 = (n+2)(n+4)/2$

OR $n(n+3)/2 = (n+2)(n+5)/2$

i.e. iff $n(n+2) = (n+2)(n+4)$

OR $n(n+3) = (n+2)(n+5)$

i.e. iff $n^2 + 2n = n^2 + 6n + 8$ **OR** $n^2 + 3n = n^2 + 7n + 10$

i.e iff $4n + 8 = 0$ **OR** $4n + 10 = 0$

i.e iff $n = -2$ **OR** $n = -5/2$ which is not possible as $n \in N$.

Hence *Equation* $K(n) = K(n+2)$ has no solution.

(iv) **To find n for which $K(n) = n^2$**

(a) Let n be *even.*

Then $K(n) = n^2$ *iff* $n(n+2)/2 = n^2$

i.e. *iff* $n^2 + 2n = 2n^2$ *Or* $n(n-2) = 0$

i.e. iff $n = 0$ *or* $n = 2$. Hence $n = 2$ is the only *even* value of n for which $K(n) = n^2$

(b) Let n be *odd.*

Then $K(n) = n^2$ *iff* $n(n+3)/2 = n^2$

i.e. *iff* $n^2 + 3n = 2n^2$ *Or* $n(n-3) = 0$

i.e. iff $n = 0$ *or* $n = 3$. Hence $n = 3$ is the only *odd* value of n for which $K(n) = n^2$

So *2* and *3* are the only *solutions* of $K(n) = n^2$

2.5 Summation and product

(i) **For n odd $\Sigma K(2n) - \Sigma K(2n-1) = K(n)$**

$$\Sigma K(2n) = \Sigma n(2n+2) = 2\Sigma n(n+1) = 2\Sigma (n^2+n)$$

$$\Sigma K(2n-1) = \Sigma (2n-1)(2n+2)/2n$$

$$= \Sigma (2n-1)(n+1) = \Sigma (2n^2+n-1)$$

$$\therefore \Sigma K(2n) - \Sigma K(2n-1) = \Sigma (n+1) = n(n+1)/2 + n$$

$$= n(n+3)/2 = K(n)$$

Hence for n odd $\Sigma K(2n) - \Sigma K(2n-1) = K(n)$

(ii)
$$\sum_{m=1}^{m=n} K(a^m) = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)$$

$$= \frac{a(a^n-1)}{2(a^2-1)}(a^{n+1}+3a+2) \text{ if } a \text{ is even}$$

$$= \frac{a(a^n-1)}{2(a^2-1)}(a^{n+1}+4a+3) \text{ if } a \text{ is odd}$$

(a) **Let a is *even***. Then

$$\sum_{m=1}^{m=n} K(a^m) = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)$$

$$= a(a+2)/2 + a^2(a^2+2)/2 + a^3(a^3+2)/2 + \ldots + a^n(a^n+2)/2$$

$$= (a^2/2 + a) + (a^4/2 + a^2) + (a^6/2 + a^3) + \ldots + (a^{2n}/2 + a^n)$$

$$= (1/2)\{a^2 + a^4 + a^6 + \ldots + a^{2n}\} + \{a + a^2 + a^3 + \ldots + a^n\}$$

$$= (1/2)\{a^2 + (a^2)^2 + (a^2)^3 + \ldots + (a^2)^n\} + \{a + a^2 + a^3 + \ldots + a^n\}$$

$$= \frac{1}{2}a^2\frac{(a^{2n}-1)}{a^2-1} + \frac{a(a^n-1)}{a-1}$$

$$= \frac{a^2}{2}\frac{(a^n-1)(a^n+1)}{(a-1)(a+1)} + \frac{a(a^n-1)}{a-1}$$

$$= \frac{a(a^n-1)}{2(a-1)}\left\{\frac{a(a^n+1)}{(a+1)} + 2\right\}$$

$$= \frac{a(a^n-1)}{2(a-1)}\left\{\frac{a^{n+1}+a+2a+2}{(a+1)}\right\}$$

$$= \frac{a(a^n - 1)}{2(a^2 - 1)} \quad (a^{n+1} + 3a + 2)$$

Hence $K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)$

$$= \frac{a(a^n - 1)}{2(a^2 - 1)} \quad (a^{n+1} + 3a + 2) \text{ if } a \text{ is } even$$

(b) **Let *a* is *odd*.** Then

$$\sum_{m=1}^{m=n} K(a^m) = K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)$$

$$= a(a+3)/2 + a^2(a^2+3)/2 + a^3(a^3+3)/2$$
$$+ \ldots + a^n(a^n+3)/2$$
$$= (1/2)\{ a^2 + 3a + a^4 + 3a^2 + a^6$$
$$+ 3a^3 + \ldots + a^{2n} + 3a^n \}$$
$$= (1/2)\{a^2 + a^4 + a^6 + \ldots + a^{2n}\}$$
$$+ \{a + a^2 + a^3 + \ldots + a^n\}$$
$$= (1/2)\{ [a^2 + (a^2)^2 + \ldots + (a^2)^n]$$
$$+ 3\{(a + a^2 + a^3 + \ldots + a^n)\}$$
$$= \frac{1}{2}\left\{a^2 \frac{(a^{2n} - 1)}{a^2 - 1} + \frac{3a(a^n - 1)}{a - 1}\right\}$$
$$= \frac{a(a^n - 1)}{2(a - 1)} \left\{\frac{a(a^n + 1)}{(a+1)} + 3\right\}$$
$$= \frac{a(a^n - 1)}{2(a - 1)} \left\{\frac{a^{n+1} + a + 3a + 3}{(a+1)}\right\}$$
$$= \frac{a(a^n - 1)}{2(a^2 - 1)} \quad (a^{n+1} + 4a + 3)$$

Hence $K(a) + K(a^2) + K(a^3) + \ldots + K(a^n)$

$$= \frac{a(a^n - 1)}{2(a^2 - 1)} \quad (a^{n+1} + 4a + 3) \text{ if } a \text{ is } odd$$

(iii) $\Pi\, K(2n) = 2^n.\ n!\ .\ (n+1)!$

$$\Pi K(2n) = \Pi\ 2n(2n+2)/2 = \Pi\ 2n(n+1)$$
$$= \Pi 2\ .\ \Pi n\ .\ \Pi(n+1)$$
$$= 2n\ .\ n!\ .\ (n+1)!$$

Hence $\Pi\ K(2n) = 2^n.\ n!\ .\ (n+1)!$

(iv) $\Pi\, K(2n-1) = (1/2^n).\ 2n!\ .n!\ (n+1)$

$$\Pi\, K(2n-1) = \Pi\ (2n-1)(2n+2)/2$$
$$= \Pi\ (2n-1)\ (n+1)$$
$$= \Pi\ (2n-1)\ (n+1)$$
$$= \Pi\ (2n-1)\ \Pi\ (n+1)$$
$$= (2n-1)!\ (n+1)!$$
$$= (1/2n).\ 2n!\ .\ n!\ .(n+1)$$

2.6 Inequalites

(i) (a) **For *even* numbers a and $b > 4$; $K(a.b) > K(a).K(b)$**

Assume that $K(a.b) \leq K(a).K(b)$

i.e $ab(ab+2)/2 \leq a(a+2)/2\ .\ b(b+2)/2$

$\therefore\ ab+2 \leq (a+2).(b+2)/2$

i.e. $ab \leq 2(a+b)$ . . . . (A)

Now as a and $b > 4$ so let $a = 4+h$, $b = 4+k$ for some

$h, k \in N$ $\therefore$ (A) $\Rightarrow (4+h)(4+k) \leq (8+2h)+(8+2k)$

i.e. $16 + 4h + 4k + hk \leq 16 + 2h + 2k$

i.e $2h + 2k + hk \leq 0$ . . . . (I)

But as h, k $\in N$, hence $2h + 2k + hk > 0$

This contradicts (I) Hence if both a and b are *even and* $a, b > 4$ then $K(a.b) > K(a).K(b)$

(b) **For odd numbers $a, b \geq 7$; $K(a.b) > K(a).K(b)$**

Let $K(a.b) \leq K(a).K(b)$

i.e $ab(ab+3)/2 \leq a(a+3)/2\ .\ b(b+3)/2$

$\therefore\ ab+3 \leq (a+3).(b+3)/2$

i.e. $2ab+6 \leq ab+3a+3b+9$

or $ab \leq 3a+3b+3$ . . . . . . (B)

Now as $a, b \geq 7$ so let $a = 7+h$, $b = 7+k$ for some $h, k \in W$

$\therefore$ (B) $\Rightarrow (7+h)(7+k) \leq 3(7+h)+3(7+k)+3$

i.e. $49 + 7h + 7k + hk \leq 45 + 3h + 3k$

i.e $4 + 4h + 4k + hk \leq 0$ . . . . . . (II)

But $h, k \in W$ hence $4 + 4h + 4k + hk > 0$

This contradicts (**II**) Hence $K(a.b) > K(a).K(b)$

(c) For a *odd*, b *even* and $a, b > 5$; $K(a.b) > K(a).K(b)$

Let $K(a.b) \leq K(a).K(b)$

i.e $ab(ab+2)/2 \leq a(a+3)/2 \;.\; b(b+2)/2$

$\therefore\; ab + 2 \leq (a+3).(b+2)\;/\;2$

i.e. $ab \leq 2a + 3b + 2 \quad . \; . \; . \; . \quad . \; . \; .$ **(C)**

Now $a, b > 5$ *so let* $a = 6 + h$ and $b = 6 + k$

for some $h, k \in W$

$\therefore$ (C) $\Rightarrow (6+h)(6+k) \leq 2(6+h) + 3(6+k) + 2$

i.e. $36 + 6h + 6k + hk \leq 12 + 2h + 18 + 3k + 2$

i.e $4h + 3k + hk + 4 \leq 0 \quad . \; . \; . \; . \quad . \; . \; .$ **(III)**

But $h, k \in W \;\therefore\; 4h + 3k + hk + 4 > 0$

This contradicts (**III**) Hence $K(a.b) > K(a).K(b)$

Note :- It follows from (xii) (a) , (b) and (c) that in general if $a, b > 5$ then $K(a.b) > K(a).K(b)$

(ii) If $a > 5$ then for all $n \in N$; $K(a^n) > n\,K(a)$

As $a > 5 \;\therefore\; K(a^n) = K(a.a.a \ldots n\ times)$

$> K(a).K(a).K(a)$ *up to n times*

$> \{K(a)\}^n \geq n\,K(a)$

Hence if $a > 5$ then for all $n \in N$; $K(a^n) > n\,K(a)$

2.7 <u>Summation of reciprocals.</u>

(i) $\displaystyle\sum_{n=1}^{n=\infty} \frac{1}{K(2n)}$ **is convergent.**

$K(2n) = 2n(2n+2)/2 = 2n(n+1)$

$$\therefore \quad \frac{1}{K(2n)} = \frac{1}{2n(n+1)} = \frac{1}{2n^2(1 + 1/n)} \leq 1/n^2$$

So *series* is *dominated* by *convergent series* and hence it is *convergent.*

(ii) $\sum_{n=1}^{n=\infty} \frac{1}{K(2n-1)}$ **is convergent.**

$K(2n-1) = (2n-1)(2n+2)/2 = (2n-1)(n+1)$

$$\therefore \quad \frac{1}{K(2n-1)} = \frac{1}{(2n-1)(n+1)}$$

$$= \frac{1}{n^2(2-1/n)(1+1/n)}$$

$$\leq 1/n^2$$

Hence by *comparison test series* is *convergent.*

(iii) $\sum_{n=1}^{n=\infty} \frac{1}{K(n)}$ **is convergent.**

$K(n) \geq n(n+2)/2$

$$\therefore \quad \frac{1}{K(n)} \leq \frac{2}{n^2(1+2/n)} \leq 1/n^2$$

Hence *series* is *convergent.*

(iv) $\sum_{n=1}^{n=\infty} \frac{K(n)}{n}$ **is divergent.**

$$\frac{K(n)}{n} \geq \frac{n+2}{2} \geq \frac{n}{2}$$

Hence series is divergent.

2.8 Limits.

(i) $\lim_{n \to \infty} \frac{K(2n)}{\sum 2n} = 2$

$K(2n) = 2n(2n+2)/2 = 2n(n+1)$

$\Sigma\, 2n = 2\Sigma\, n = n(n+1)$

$$\frac{K(2n)}{\sum 2n} = \frac{2n(n+1)}{n(n+1)} = 2$$

$$\therefore \lim_{n \to \infty} \frac{K(2n)}{\sum 2n} = 2$$

(ii) $$\lim_{n\to\infty} \frac{K(2n-1)}{\sum(2n-1)} = 2$$

$$K(2n-1) = (2n-1)(2n-1+3)/2$$

$$= (2n-1)(2n+2)/2 = (2n-1)(n+1)$$

$$\Sigma\, 2n-1 = 2n(n+1)/2 - n = n^2$$

$$\therefore \frac{K(2n-1)}{\sum(2n-1)} = \frac{(2n-1)(n+1)}{n^2} = (2-\frac{1}{n})(1+\frac{1}{n})$$

$$\therefore \lim_{n\to\infty} \frac{K(2n-1)}{\sum(2n-1)} = 2$$

(iii) $$\lim_{n\to\infty} \frac{K(2n+1)}{K(2n-1)} = 1$$

$$K(2n+1) = (2n+1)(2n+1+3)/2$$

$$= (2n+1)(n+2)$$

$$K(2n-1) = (2n-1)(2n-1+3)/2$$

$$= (2n-1)(2n+2)/2 = (2n-1)(n+1)$$

$$\therefore \frac{K(2n+1)}{K(2n-1)} = \frac{(2n+1)(n+2)}{(2n-1)(n+1)}$$

OR $$\frac{K(2n+1)}{K(2n-1)} = \frac{(2+\frac{1}{n})(1+\frac{2}{n})}{(2-\frac{1}{n})(1+\frac{1}{n})}$$

$$\therefore \lim_{n\to\infty} \frac{K(2n+1)}{K(2n-1)} = 1$$

(iv) $$\lim_{n\to\infty} \frac{K(2n+2)}{K(2n)} = 1$$

$$K(2n+2) = (2n+2)(2n+2+2)/2$$

$$= 2(n+1)(n+2)$$

$$K(2n) = 2n(2n+2)/2 = 2n(n+1)$$

$$\therefore \frac{K(2n+2)}{K(2n)} = \frac{2(n+1)(n+2)}{2n(n+1)}$$

OR $$\frac{K(2n+2)}{K(2n)} = (1+\frac{2}{n})$$

$$\therefore \lim_{n\to\infty} \frac{K(2n+2)}{K(2n)} = 1$$

2.9 Additional Properties.

(i) **Let *C* be the *continued fraction* of the *sequence {K(n)}***

$$C = K(1) + \cfrac{K(2)}{K(3) + \cfrac{K(4)}{K(5) + \cfrac{K(6)}{K(7) + \; . \; . \; .}}}$$

$$= K(1) + \frac{K(2)}{K(3)+}\;\frac{K(4)}{K(5)+}\;\frac{K(6)}{K(7)+}\;. \; . \; .$$

The n th term $T_n = \dfrac{K(2n)}{K(2n+1)} = \dfrac{2n^2 + 2n}{2n^2 + 5n + 2}$

Hence $T_n < 1$ for all n and $\therefore$ with respect to [3], C is convergent and $2 < C < 3$.

(ii) ***K (2 ^{n} – 1) + 1* is a *triangular number.***

Let $x = 2n$ then

$$K(2n-1) + 1 = K(x-1) + 1$$

$$= \{ (x-1)(x+2)/2 \} + 1$$

$$= \{ x^2 + x \}/2$$

$$= x(x+1)/2 \text{ which is a triangular number.}$$

(iii) ***Fibonacci sequence does not exist in the sequence { K(n) }***

(a) If possible then let $K(n) + K(n+1) = K(n+2)$ for some n where n is *even*.

$$\therefore\; n(n+2)/2 + (n+1)(n+4)/2 = (n+2)(n+4)/2$$

$$\therefore\; (n^2 + 2n) + (n^2 + 5n + 4) = n^2 + 6n + 8$$

$$\therefore\; n^2 + n - 4 = 0 \;\; \textbf{OR} \;\; n = \frac{-1 \pm \sqrt{17}}{2}$$ which is not possible as $n \in N$.

(b) Let $K(n) + K(n+1) = K(n+2)$ for some n where n is *odd*.

$$\therefore\; n(n+3)/2 + (n+1)(n+3)/2 = (n+2)(n+5)/2$$

$$\therefore\; (n+3)\;(2n+1) = n^2 + 7n + 10$$

$$\therefore\; n^2 = 7 \;\; \textbf{OR} \;\; n = \sqrt{7}$$ which is not possible as $n \in N$.

Hence there is no *Fibonacci sequence* in *{ K(n)}*

Similarly there is no *Lucas sequence* in *{ K (n) }*

(iv) $K(n) > max\{K(d)$: *Where d is a proper divisor of n and n is composite* $\}$.

As d is a *proper divisor* of n $\therefore$ $d < n$ and as *function* K is *strictly monotonic increasing* hence $K(d) < K(n)$.

So for each proper divisor d we have $K(n) > K(d)$

and hence $K(n) > max\{K(n)\}$

(v) ***Palindromes in*** $\{K(n)\}$

$K(11) = 77$, $K(21) = 252$, $K(29) = 464$,

$K(43) = 989$, $K(64) = 212$

are only *Palindromes* for $n \leq 100$.

(vi) ***Pythagorean Triplet***

We know that $(5, 12, 13)$ is a *Pythagorean Triplet*.

Similarly $(K(5), K(12), K(13))$ is a Linear Triplet because

$K(5) + K(12) = K(13)$.

(vii) $K(2^n) = 2^n(2^n+2)/2 = 2^{2n-1} + 2^n$

$\therefore K(2^3) = 2^5 + 2^3 = 32 + 8 = 40$ and $40 + 1 = 41$ is *prime.*

Similarly $K(2^4) = 2^7 + 2^4 = 128 + 16 = 144$ and $140 - 1 = 139$ is *prime.*

Hence it is *conjectured* that $K(2^n) - 1$ *or* $K(2^n) + 1$ *is prime.*

3.1 To find K^{-1} when n is odd

$\therefore$ $K(n) = n(n+3)/2 = t$ (*say*)

$\therefore$ $n = K^{-1}(t)$ *Also as* $n(n+3)/2 = t$

$$\therefore n = \frac{-3+\sqrt{9+8t}}{2} \quad \textbf{OR} \quad K^{-1}(t) = n = \frac{-3+\sqrt{9+8t}}{2}$$

$$\textbf{OR} \quad K^{-1}(t_r) = \frac{-3+\sqrt{9+8t_r}}{2} = n_r$$

Note:

(I) In the above expression plus sign is taken to ensure that

$K^{-1}(t_r) \in N$.

(II) Also $K^{-1}(t_r) \in N$ *iff* $\sqrt{9 + 8t_r}$ is an *odd integer.*

and for this $9 + 8t_r$ should be a perfect square.

From above two observations we get possible values of t_r as $2, 9, 20, 35$ *etc* . . .

3.2 **Following are some examples of $K^{-1}(t_r)$**

r	t_r	$K^{1}(t_r) = n_r$	$q_r = t_r / n_r$
1	2	1	2
2	9	3	3
3	20	5	4
4	35	7	5
5	54	9	6
6	77	11	7
7	104	13	8

3.3 **Following results are obvious.**

(i) $K^{1}(t_r) = n_r = 2r - 1$

(ii) $t_r = t_{r-1} + (4r - 1)$

(iii) $t_r = n_r q_r = (2r - 1)\, q_r$

(iv) $n_r = q_r + (r - 2)$

(v) $\Sigma t_r = \Sigma t_{r-1} + r \,.\, n_r$

(vi) Every t_{r+1} is a *triangular number.*

(vii) As $t_r - t_{r-1} = 4r - 1$

$\therefore$ *Second diffierence* $D^2(t_r) = 4r - 1 - [4(r-1) - 1] = 4$

3.4 **To find K^{-1} when n is even**

$$\therefore \; K(n) = n(n+2)/2 = t \;(say)$$

$$\therefore \; n = K^{-1}(t) \; Also \; as \; n(n+2)/2 = t$$

$$\therefore \; n = \frac{-2+\sqrt{4+8t}}{2} \quad OR \quad K^{-1}(t) = n = -1+\sqrt{1+2t}$$

$$OR \quad K^{-1}(t_r) = -1+\sqrt{1+2t_r} = n_r$$

Note:

(I) In the above expression plus sign is taken to ensure that

$K^{-1}(t_r) \in N.$

(II) Also $K^{-1}(t_r) \in N$ *iff* $\sqrt{1+2t_r}$ is an *odd integer.*

and for this first of all $1 + 2t_r$ should be a perfect square.

of some odd integer.

From above two observations we get possible values of t_r

as $4, 12, 24, 40$ etc . . .

3.5 **Following are some examples of $K^{-1}(t_r)$**

r	t_r	$K^{1}(t_r) = n_r$	$q_r = t_r / n_r$
1	4	2	2
2	12	4	3
3	24	6	4
4	40	8	5
5	60	10	6
6	84	12	7
7	112	14	8

3.6 **Following results are obvious.**

(i) $K^{1}(t_r) = n_r = 2r$

(ii) $t_r = t_{r-1} + 4r$

(iii) $t_r = n_r q_r = 2r \,.\, q_r$

(iv) $n_r = q_r + (r-1)$

(v) $\Sigma t_r = \Sigma t_{r-1} + (r+1) \,.\, n_r$

(vi) $t_r = n_r \, [n_r - r + 1]$

(vi) Every t_r *is a multiple of 4*

(vii) $t_r = 4p$ where p is a *triangular number.*

(viii) For $r = 8$, $t_r = 144$, $n_r = 16$ and $q_r = 9$. So for $r = 8$; t_r, n_r, and q_r

are all *perfect square.*

(ix) As $t_r - t_{r-1} = 4r$

$\therefore$ *Second diffterence* $D^2(t_r) = 4r - [4(r-1)] = 4$

3.7 Monoid

Let $M = \{K^1(2),\ K^1(4),\ K^{-1}(9),\ K^{-1}(12)\ \ldots\}$ be the collection of *images* of K^1 including both *even* and *odd* n.

Let $\bullet$ stands for multiplication. Then $(M, \bullet)$ is a *Monoid.*

For it satisfies (I) *Closure* (II) *Associativity* (III) *Identity*

Here *identity* is $K^{-1}(2)$.

In fact $(M, \bullet)$ is a *Commutative Monoid.*

As *inverse* of an element does not exist in M hence it is not a *group*.

Coincidently, *M* happens to be a *cyclic monoid* with operation + .

Because $K^1(9) = [K^1(2)]^3$

References :-

[1] **Ashbacher C :** ***Introduction to Smarandache Functions.*** **(Journal of Recreational Mathematics 1996, P. 249)**

[2] **David Gorski :** ***The Pseudo Smarandache Functions.*** **(*Smarandache Notion Journal* Vol. 12, 2000, P. 140)**

[3] **Castrillo Jose :** ***Smarandache Continued Fractions.*** **(*Smarandache Notion Journal* Vol. 9, 1998, P. 40)**

Appendix – [A]

Values of K (n) for n = 1 To 100

n	Σn	k	K(n)
1	1	1	2
2	3	1	4
3	6	3	9
4	10	2	12
5	15	5	20
6	21	3	24
7	28	7	35
8	36	4	40
9	45	9	54
10	55	5	60
11	66	11	77
12	78	6	84
13	91	13	104
14	105	7	112
15	120	15	135
16	136	8	144
17	153	17	170
18	171	9	180
19	190	19	209
20	210	10	220
21	231	21	252
22	253	11	264
23	276	23	299
24	300	12	312
25	325	25	350

n	Σn	k	K(n)
26	351	13	364
27	378	27	405
28	406	14	420
29	435	29	464
30	465	15	480
31	496	31	527
32	528	16	544
33	561	33	594
34	595	17	612
35	630	35	665
36	666	18	684
37	703	37	740
38	741	19	760
39	780	39	819
40	820	20	840
41	861	41	902
42	903	21	924
43	946	43	989
44	990	22	1012
45	1035	45	1080
46	1081	23	1104
47	1128	47	1175
48	1176	24	1200
49	1225	49	1274
50	1275	25	1300

n	Σn	k	K(n)
51	1326	51	1377
52	1378	26	1404
53	1431	53	1484
54	1485	27	1512
55	1540	55	1595
56	1596	28	1624
57	1653	57	1710
58	1711	29	1740
59	1770	59	1829
60	1830	30	1860
61	1891	61	1952
62	1953	31	1984
63	2016	63	2079
64	2080	32	2112
65	2145	65	2210
66	2211	33	2244
67	2278	67	2345
68	2346	34	2380
69	2415	69	2484
70	2485	35	2520
71	2556	71	2627
72	2628	36	2664
73	2701	73	2774
74	2775	37	2812
75	2850	75	2925

n	Σn	k	K(n)
76	2926	38	2964
77	3003	77	3080
78	3081	39	3120
79	3160	79	3239
80	3240	40	3280
81	3321	81	3402
82	3403	41	3444
83	3486	83	3569
84	3570	42	3612
85	3655	85	3740
86	3741	43	3784
87	3828	87	3915
88	3916	44	3960
89	4005	89	4094
90	4095	45	4140
91	4186	91	4277
92	4278	46	4324
93	4371	93	4464
94	4465	47	4512
95	4560	95	4655
96	4656	48	4704
97	4753	97	4850
98	4851	49	4900
99	4950	99	5049
100	5050	50	5100