On a characterization of the uniform repartition

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An important role in the theory of the hi-square criterion is played by the following fact: *if* $x_1, x_2, \ldots, x_n$ are independent random variables with Gauss distribution $N(0, \delta^2)$, then the distribution of the central statistic hi-square $\chi^2 = \sum_{i=1}^{n} (x_i + a_i)^2$ depends on $a_1, a_2, \ldots, a_n$ only by mean of the parameter $\sum_{i=1}^{n} a_i^2$. In the paper [1] one proves that this property is characteristic for the normal distribution of probability. The aim of this paper is to give a characterization of the uniform distribution of probability by mean of the hi-square statistic.

Theorem 1 *Let* $x_1, x_2, \ldots, x_n$ *independent and equally distributed random variables, where* $n \geq 2$, *then the necessary and sufficient condition that the statistic distribution* $\chi^2 = \sum_{i=1}^{n} (x_i + a_i)^2$ *depend on* $\sum_{i=1}^{n} a_i^2$ *with* $a_i \in \mathbb{R}$ *is that* x_i *be uniformly distributed.*

Proof. We define the function:

$$\psi(a) = Ee^{-(x_i+a)^2}. \tag{1}$$

It is obvious that $\psi(a) > 0$ and ψ is derivable in every $a \in \mathbb{R}$.

Using the conditions of the theorem we have

$$Ee^{-\sum_{i=1}^{n}(x_i+a_i)^2} = \prod_{i=1}^{n} Ee^{-(x_i+a_i)^2} = \prod_{i=1}^{n} \psi(a_i) = \Phi\left(\sum_{i=1}^{n} a_i\right). \tag{2}$$

Let $h(a) = \log \psi(a)$ and $H(a) = \log \Phi(a)$. From (2) we obtain:

$$\sum_{i=1}^{n} h(a_i) = H\left(\sum_{i=1}^{n} a_i\right). \tag{3}$$

If we differentiate twice the both sides of (3) by a_1, then by a_2, we obtain for every $a_1, a_2, \ldots, a_n$:

$$H''\left(\sum_{i=1}^{n} a_i\right) = 0. \tag{4}$$

In this way

$$H(a) = C_1 a + C_2 . \tag{5}$$

From (1) and (3) we obtain:

$$\psi(a) = \int e^{-(x+a)^2} dF(x) = e^{C_1 a + C_3} \tag{6}$$

where $F(x) = P(x_i < x)$.

In the following step we consider the substitution:

$$e^{-x^2} dF(x) = dG. \tag{7}$$

In this case (6) can be written in the form:

$$\int e^{-2ax} dG(x) = e^{C_4 a + C_5}. \tag{8}$$

It follows, using the uniqueness theorem for the Laplace transformation, that $dG = C_5 \Delta(x - C_6)$ for every C_5 and C_6, where Δ is the Dirac function. Using again relation (7), it follows that F is the distribution function of the uniform random variable.

The sufficiency can be proved by a straightforward verification.

References

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[3] Cagan A.M., Zingher A.A. *Sample mean as an estimator of location parameter. Case of non quadratic lass function* Sankhga Ser.A33(1971) 351-358.

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