ON A FUNCTION IN THE NUMBERS THEORY

by

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Abstract. In the present paper we study some series concerning the following function of the Numbers Theory [1]: "S : N->N such that S(n) is the smallest k with property that k! is divisible by n".

1. Introduction. The following functions in Numbers Theory are well - known : the function $\mu(n)$ of Môbius, the function $\xi(s)$ of Riemann ($\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, s=\sigma+it\in C$), the function $\Lambda(n)$ of Mangoldt $\left(\Lambda(n)=\begin{cases}\log p, \text{if } n=p^m\\ o, \text{if } n\neq p^m\end{cases}\right)$ etc.

The purpose of this paper is to study some series concerning the following function of the Numbers Theory "[1] S : N->N such that S(n) is the smallest integer k with the propriety that k! is divisible by n".

We first prove the divergence of some series involving the S function, using an unitary method, and then we prove that the series $\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)...S(n)}$ is convergent to a number S $\in$ (71/100, 101/100) and we study some applications of this series in the Numbers Theory .

Then we prove that series $\sum_{n=2}^{\infty}\frac{1}{S(n)!}$ is convergent to a real numbers s $\in$(0.717, 1.253) and that the sum of the remarkable series $\sum_{n\geq 2}\frac{S(n)}{n!}$ is a irrational number.

2. The main results

Proposition 1. If $(x_n)_{n\geq 1}$ is strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty}\frac{x_{n+1}-x_n}{S(x_n)}, \tag{1}$$

is divergent.

Proof. We consider the function $f:[x_n, x_{n+1}]\to\mathbf{R}$, defined by $f(x)=\ln\ln x$ is meets the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n\in(x_n, x_{n+1})$ such that :

$$\ln\ln x_{n+1}-\ln\ln x_n=\frac{1}{c_n\ln c_n}(x_n-x_{n+1}). \tag{2}$$

Because $x_n<c_n<x_{n+1}$, we have :

$$\frac{x_{n+1}-x_n}{x_{n+1}\ln x_{n+1}}<\ln\ln x_{n+1}-\ln\ln x_n<\frac{x_{n+1}-x_n}{x_n\ln x_n}, (\forall)n\in\mathbf{N}, \tag{3}$$

if $x_n\neq 1$.

We know that for each $n\in\mathbf{N}^*\setminus\{1\}$, $\frac{S(n)}{n}\leq 1$, i.e.

$$0<\frac{S(n)}{n\ln n}\leq\frac{1}{\ln n}, \tag{4}$$

from where it results that $\lim_{n\to\infty}\frac{S(n)}{n\ln n}=0$. Hence there is k>0 such that $\frac{S(n)}{n\ln n}<k$, i.e., $n\ln n>\frac{S(n)}{k}$ for any $n\in N^*$, so

$$\frac{1}{x_n\ln x_n}<\frac{k}{S(x_n)}. \tag{5}$$

Introducing (5) in (3) we obtain :

$$\ln\ln x_{n+1}-\ln\ln x_n<k\frac{x_{n+1}-x_n}{S(x_n)},\ (\forall)n\in N^*\setminus\{1\}. \tag{6}$$

Summing up after n it results :

$$\sum_{n=1}^{m}\frac{x_{n+1}-x_n}{S(x_n)}>\frac{1}{k}(\ln\ln x_{m+1}-\ln\ln x_1).$$

Because $\lim_{m\to\infty}x_m=\infty$ we have $\lim_{m\to\infty}\ln\ln x_m=\infty$, i.e., the series :

$$\sum_{n=1}^{\infty}\frac{x_{n+1}-x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

Proposition 2. Series $\sum_{n=2}^{\infty}\frac{1}{S(n)}$ is divergent.

Proof. We use Proposition 1 for $x_n = n$.

Remarks.

1) If x_n is the n - the prime number, then the series $\sum_{n=1}^{\infty}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent.

2) If the sequence $(x_n)_{n\geq 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent.

3) The series $\sum_{n=1}^{\infty}\frac{1}{S(2n+1)}, \sum_{n=1}^{\infty}\frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent forms are divergent.

Proposition 3. The series

$$\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)\ldots S(n)}$$ is convergent to a number $s\in(71/100, 101/100)$.

Proof. From the definition it results $S(n)\leq n!$, $(\forall)n\in N^*\setminus\{1\}$, so $\frac{1}{S(n)}\geq\frac{1}{n!}$.

Summing up, beginning with n=2 we obtain :

$$\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)\ldots S(n)}\geq\sum_{n=2}^{\infty}\frac{1}{n!}=e-2.$$

The product S(2) S(3)...S(n) is greater than the product of prime numbers from the set {1, 2, ..., n}, because S(p)=p, for p= prime number. Therefore :

$$\frac{1}{\prod_{i=2}^{n}S(i)}<\frac{1}{\prod_{i=2}^{n}p_i}, \tag{7}$$

where p_k is the biggest number smaller or equal to n.

There are the inequalities :

$$S=\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)...S(n)}=\frac{1}{S(2)}+\frac{1}{S(2)S(3)}+\frac{1}{S(2)S(3)S(4)}+....+$$
$$+\frac{1}{S(2)S(3)..S(k)}+...<\frac{1}{2}+\frac{2}{2\cdot 3}+\frac{2}{2\cdot 3\cdot 5}+\frac{2}{2\cdot 3\cdot 5\cdot 7}+$$
$$+\frac{2}{2\cdot 3\cdot 5\cdot 7\cdot 11}+....+\frac{p_{k+1}-p_k}{p_1p_2...p_k}+.... \quad (8)$$

Using the inequality $p_1p_2...p_k>p_{k+1}^3, (\forall)k\geq 5$[5], we obtain :

$$S<\frac{1}{2}+\frac{1}{3}+\frac{1}{15}+\frac{2}{105}+\frac{1}{p_6^2}+\frac{1}{p_7^2}+....+\frac{1}{p_{k+1}^2}+... \quad (9)$$

We symbolise by $P=\frac{1}{p_6^2}+\frac{1}{p_7^2}+...$ and observe that $P<\frac{1}{13^2}+\frac{1}{14^2}+\frac{1}{15^2}+...$

It results :

$$P<\frac{\pi^2}{6}-\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{12^2}\right),$$

where

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+... \text{ (EULER).}$$

Introducing in (9) we obtain :

$$S<\frac{1}{2}+\frac{1}{3}+\frac{1}{15}+\frac{1}{105}+\frac{\pi^2}{6}-1-\frac{1}{2^2}-\frac{1}{3^2}-...-\frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71<\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)...S(n)}<0,79. \quad (10)$$

The proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right ranging :

$$\sum_{n=2}^{\infty}\frac{1}{S(2)S(3)...S(n)}<0,97. \quad (11)$$

Proposition 4. Let α be a fixed real number, $\alpha\geq 1$. Then the series $\sum_{n=2}^{\infty}\frac{n^\alpha}{S(2)S(3)...S(n)}$ is convergent.

Proof. Be $(x_k)_{k\geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)}=\frac{2^\alpha}{2}=2^{\alpha-1}$$
$$\frac{3^\alpha}{S(2)S(3)}=\frac{3^\alpha}{p_1p_2}$$
$$\frac{4^\alpha}{S(2)S(3)S(4)}<\frac{4^\alpha}{p_1p_2}<\frac{p_3^\alpha}{p_1p_2}$$
$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)}<\frac{5^\alpha}{p_1p_2p_3}<\frac{p_4^\alpha}{p_1p_2p_3}$$
$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)}<\frac{6^\alpha}{p_1p_2p_3}<\frac{p_4^\alpha}{p_1p_2p_3}$$

....

$$\frac{n^\alpha}{S(2)S(3)...S(n)}<\frac{n^\alpha}{p_1p_2...p_k}<\frac{p_{k+1}^\alpha}{p_1p_2...p_k},$$

where $p_i\leq n, i\in\{1,...,k\}, p_{k+1}>n$.

Therefore

$$\sum_{n=2}^{\infty}\frac{n^{\alpha}}{S(2)S(3)...S(n)}<2^{\alpha-1}+\sum_{n=2}^{\infty}\frac{(p_{k+1}-p_k)\cdot p_{k+1}^{\alpha}}{S(2)S(3)...S(n)}<$$

$$<2^{\alpha-1}+\sum_{n=2}^{\infty}\frac{p_{k+1}^{\alpha}}{p_1p_2...p_k}.$$

Then it exists $k_0\in\mathbf{N}$ such that for any $k\geq k_0$ we have :

$p_1p_2...p_k>p_{k+1}^{\alpha+3}$.

Therefore

$$\sum_{n=2}^{\infty}\frac{n^{\alpha}}{S(2)S(3)...S(n)}<2^{\alpha-1}+\sum_{k=1}^{k_0-1}\frac{p_{k+1}^{\alpha+1}}{p_1p_2...p_k}+\sum_{k\geq k_0}\frac{1}{p_{k+1}^2}.$$

Because the series $\sum_{k\geq k_0}\frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent too .

Consequence 1. It exists $n_0\in\mathbf{N}$ so that for each $n\geq n_0$ we have $S(2)S(3)...S(n)>n^{\alpha}$.

Proof. Because $\lim_{n\to\infty}\frac{n^{\alpha}}{S(2)S(3)...S(n)}=0$, there is $n_0\in\mathbf{N}$ so that $\frac{n^{\alpha}}{S(2)S(3)...S(n)}<1$ for each $n\geq n_0$.

Consequence 2. It exists $n_0\in\mathbf{N}$ so that :

$S(2)+S(3)+...+S(n)>(n-1)n^{\frac{\alpha}{n-1}}$ for each $n\geq n_0$.

Proof. We apply the inequality of averages to the numbers S(2), S(3), ..., S(n) :

$$S(2)+S(3)+...+S(n)>(n-1)\sqrt[n-1]{S(2)S(3)...S(n)}>(n-1)n^{\frac{\alpha}{n-1}},\ \forall n\geq n_0.$$

We can write it as it follows :

$\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{3!}+...=\frac{1}{2!}+\frac{2}{3!}+\frac{4}{4!}+\frac{8}{5!}+\frac{14}{6!}+...=\sum_{n=2}^{\infty}\frac{a(n)}{n!}$, where a(n) is the number of solutions for the equation $S(x)=n$, $n\in\mathbf{N}$, $n\geq 2$.

It results from the equality S(x)=n that x is a divisor of n!, so a(n) is smaller than d(n!).

So, $a(n)<d(n!)$.

Lemma 1. We have the inequality :

$d(n)\leq n-2$, for each $n\in\mathbf{N}$, $n\geq 7$. (12)

Proof. Be $n=p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ with $p_1, p_2, ..., p_k$ prime numbers, and $a_i\geq 1$ for each $i\in\{1,2,...,k\}$. We consider the function $f:[1,\infty)\to\mathbf{R}$, $f(x)=a^x-x-2$, $a\geq 2$, fixed. It is derivable on $[1,\infty)$ and $f'(x)=a^x\ln a-1$. Because $a\geq 2$, and $x\geq 1$ it results that $a^x\geq 2$, so $a^x\ln a\geq 2\ln a=\ln a^2\geq\ln 4>\ln e=1$, $f'(x)>0$ for each $x\in[1,\infty)$ and $a\geq 2$, fixed. But $f(1)=a-3$. It results that for $a\geq 3$ we have $f(x)\geq 0$ means $a^x\geq x+2$.

Particularly, for $a=p_i, i\in\{1,2,..,k\}$, we obtain $p_i^{a_i}\geq a_i+2$ for each $p_i\geq 3$.

If $n=2^s$, $s\in\mathbf{N}^*$, then $d(n)=s+1<2^s-2=n-2$ for $s\geq 3$.

So we can assume $k\geq 2$, i.e. $p_2\geq 3$. The following inequalities result :

$p_1^{a_1}\geq a_1+1$

$p_2^{a_2}\geq a_2+1$

....

$p_k^{a_k}\geq a_k+1$,

equivalent with

$p_1^{a_1}\geq a_1+1, p_2^{a_2}-1\geq a_2+1, ..., p_k^{a_k}-1\geq a_k+1.$ (13)

By multiplying, member with member, of the inequalities (13) we obtain :

$$p_1^{a_1}(p_2^{a_2}-1)\ldots(p_k^{a_k}-1) \geq (a_1+1)(a_2+1)\ldots(a_k+1) = d(n). \qquad (14)$$

Considering the obvious inequality :

$$n-2 \geq p_1^{a_1}(p_2^{a_2}-1)\ldots(p_k^{a_k}-1) \qquad (15)$$

and using (14) it results that :

n-2 ≥ d(n) for each n ≥ 7.

Lemma 2. d(n!) < (n-2)! for each n ∈**N**, n ≥ 7. (16)

Proof. We carry out an induction after n. So, for n=7,

$$d(7!) = d(2^4 \cdot 2^3 \cdot 5 \cdot 7) = 60 < 120 = 5!.$$

We assume that d(n!) < (n-2)!.

$$d((n+1)!) = d(n!(n+1)) < d(n!)\,d(n+1) < (n-2)!d(n+1) < (n-2)!(n-1) = (n-1)!,$$

because in according to Lemma 1, d(n+1) < n-1.

Proposition 5. The series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a number s ∈ (0.717, 1.253).

Proof. From Lemma 2 it results that a(n) < (n-2)!, so $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$ for every n∈ **N**, n ≥ 7 and $\sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{6} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)}$.

Therefore $\sum_{n=2}^{\infty} \frac{a(n)}{n!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2-n}$. (17)

Because $\sum_{n=2}^{\infty} \frac{1}{n^2-n} = 1$ we have there is a number s > 0, $s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}$.

From (17) we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2-2} - \frac{1}{3^2-3} - \frac{1}{4^2-4} +$$
$$+ \frac{1}{5^2-5} + \frac{1}{6^2-6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253.$$

But, because S(n) ≤ n for every n ∈**N***, it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - \dot{2}.$$

Consequently, for the number s we obtain the range e-2 < s < 1,253, i.e., 0,717 < s < 1,253.

Because S(n) ≤ n, it results $\sum_{n\geq 2} \frac{S(n)}{n!} \leq \sum_{n\geq 2} \frac{1}{(n-1)!}$. Therefore the series $\sum_{n\geq 2} \frac{S(n)}{n!}$ is convergent to a number f.

Proposition 6. The sum f of the series $\sum_{n\geq 2} \frac{S(n)}{n!}$ is an irrational number.

Proof. From the above results that $\lim_{n\to\infty} \sum_{i=2}^{n} \frac{S(n)}{n!} = 1$. Under these circumstances that $f \in Q, f > 0$. Therefore it exists a,b ∈**N**, (a,b)=1, so that $f=\frac{a}{b}$.

Let p be a fixed prime number, p > b, p ≥ 3. Obviously, $\frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i\geq p} \frac{S(i)}{i!}$ which leads to :

$$\frac{(p-1)!a}{b} = \sum_{1=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i \geq p} \frac{(p-1)!S(i)}{i!}.$$

Because p > b results that $\frac{(p-1)!a}{b} \in \mathbf{N}$ and $\sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in \mathbf{N}$. Consequently we have $\sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbf{N}$ too.

Be $\alpha = \sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbf{N}$. So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} + \frac{(p-1)!S(p+2)}{(p+2)!} + \dots$$

Because p is a prime number it results S(p)=p.

So

$$\alpha = 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)!} + \dots > 1 \qquad (18)$$

We know that $S(p+1) \leq p + 1 (\forall) i \geq 1$, with equality only if the number p+i is prime. Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \dots < 1 + 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots = \frac{p}{p-1} < 2 \qquad (19)$$

From the inequalities (18) and (19) results that $1<\alpha<2$, impossible, because $\alpha \in \mathbf{N}$. The proposition is proved.

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