ON A PROBLEM OF F. SMARANDACHE*

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ABSTRACT. Let $d_s(n)$ denotes the sum of the base 10 digits of $n \in N$. For natural $x \geq 2$ and arbitrary fixed exponent $m \in N$, let $A_m(x) = \sum_{n<x} d_s^m(n)$. The main purpose of this paper is to give two exact calculating formulas for $A_1(x)$ and $A_2(x)$.

1. INTRODUCTION

For any positive integer n, let $d_s(n)$ denotes the sum of the base 10 digits of n. For example, $d_s(0) = 0$, $d_s(1) = 1$, $d_s(2) = 2, \cdots$, $d_s(11) = 2$, $d_s(12) = 3, \cdots\cdots$. In problem 21 of book [1], Professor F.Smaradache ask us to study the properties of sequence $\{d_s(n)\}$. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in N$, let

$$A_m(x) = \sum_{n<x} d_s^m(n). \tag{1}$$

The main purpose of this paper is to study the calculating problem of $A_m(x)$, and use elementary methods to deduce two exact calculating formulas for $A_1(x)$ and $A_2(x)$. That is, we shall prove the following:

Theorem. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + \cdots + a_s 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, \cdots, s$. Then we have the calculating formulas*

$$A_1(x) = \sum_{i=1}^{s} a_i \cdot \left(\frac{9}{2}k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2} \right) \cdot 10^{k_i};$$

$$A_2(x) = \sum_{i=1}^{s} a_i \cdot \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6} \right] \cdot 10^{k_i}$$
$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j} \right] \cdot \left(\sum_{j=1}^{i-1} a_j \right).$$

For general integer $m \geq 3$, using our methods we can also give an exact calculating formula for $A_m(x)$. But in these cases, the computations are more complex.

Key words and phrases. F.Smarandache problem; Sum of base 10 digits; Calculating formula.
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2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. *For any integer $k \geq 0$, we have the identities*

a) $A_1(10^k) = \frac{9}{2} \cdot k \cdot 10^k$;

b) $A_1(a \cdot 10^k) = \left(\frac{9}{2}k + \frac{a-1}{2}\right) \cdot a \cdot 10^k, \quad 1 \leq a \leq 9.$

Proof. We first prove a) of Lemma 1 by induction. For $k = 0$ and 1, we have $A_1(10^0) = A_1(1) = 0$, $A_1(10^1) = A_1(10) = 45$. So that the identity

$$A_1(10^k) = \sum_{n<10^k} d_s(n) = \frac{9}{2} \cdot k \cdot 10^k \tag{2}$$

holds for $k = 0$ and 1. Assume (2) is true for $k = m - 1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(10^m) &= \sum_{n<9\cdot 10^{m-1}} d_s(n) + \sum_{9\cdot 10^{m-1} \leq n < 10^m} d_s(n) \\
&= A_1(9 \cdot 10^{m-1}) + \sum_{0 \leq n < 10^{m-1}} d_s(n + 9 \cdot 10^{m-1}) \\
&= A_1(9 \cdot 10^{m-1}) + \sum_{0 \leq n < 10^{m-1}} (d_s(n) + 9) \\
&= A_1(9 \cdot 10^{m-1}) + 9 \cdot 10^{m-1} + \sum_{n<10^{m-1}} d_s(n) \\
&= A_1(9 \cdot 10^{m-1}) + 9 \cdot 10^{m-1} + A_1(10^{m-1}) \\
&= A_1(8 \cdot 10^{m-1}) + (8+9) \cdot 10^{m-1} + 2A_1(10^{m-1}) \\
&= \cdots\cdots\cdots \\
&= (1+2+3+4+5+6+7+8+9) \cdot 10^{m-1} + 10A_1(10^{m-1}) \\
&= \frac{9}{2} \cdot 10^m + 10 \cdot \frac{9}{2} \cdot (m-1) \cdot 10^{m-1} \\
&= \frac{9}{2} \cdot m \cdot 10^m.
\end{aligned}$$

That is, (2) is true for $k = m$. This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned}
A_1(a \cdot 10^k) &= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{(a-1)\cdot 10^k \leq n < a \cdot 10^k} d_s(n) \\
&= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{0 \leq n < 10^k} d_s(n + (a-1) \cdot 10^k) \\
&= \sum_{n<(a-1)\cdot 10^k} a(n) + (a-1) \cdot 10^k + \sum_{n<10^k} d_s(n) \\
&= A_1((a-1) \cdot 10^k) + (a-1) \cdot 10^k + A_1(10^k).
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any integer $k \geq 0$ and $1 \leq a \leq 9$, we have the identities*

c) $A_2(10^k) = \dfrac{81k+33}{4} \cdot k \cdot 10^k$;

d) $A_2(a \cdot 10^k) = \left[\dfrac{k(81k+33)}{4} + \dfrac{9k}{2}(a-1) + \dfrac{(a-1)(2a-1)}{6}\right] \cdot a \cdot 10^k.$

Proof. These results can be deduced by Lemma 1, induction and the recurrence formula

$$\begin{aligned}
A_2(10^{k+1}) &= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{9\cdot 10^k \leq n < 10^{k+1}} d_s^2(n) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} d_s^2(n+9\cdot 10^k) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} (d_s(n)+9)^2 \\
&= A_2(9\cdot 10^k) + 9^2\cdot 10^k + 18A_1(10^k) + A_2(10^k) \\
&= \cdots\cdots\cdots \\
&= 10A_2(10^k) + (1^2+2^2+\cdots+9^2)\cdot 10^k + 2\cdot(1+2+\cdots+9)A_1(10^k) \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + 90\cdot\frac{9}{2}\cdot k\cdot 10^k \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + \frac{81}{2}\cdot k\cdot 10^{k+1}.
\end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer x, let $x = a_1\cdot 10^{k_1} + a_2\cdot 10^{k_2} + \cdots + a_s\cdot 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ under the base 10. Then applying Lemma 1 repeatedly we have

$$\begin{aligned}
A_1(x) &= \sum_{n<a_1\cdot 10^{k_1}} d_s(n) + \sum_{a_1\cdot 10^{k_1}\leq n<x} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n+a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} (d_s(n)+a_1) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + A_1(x-a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + A_1(a_2\cdot 10^{k_2}) + a_1(x-a_1\cdot 10^{k_1}) \\
&\quad + a_2(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) + A_1(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) \\
&= \cdots\cdots\cdots
\end{aligned}$$

$$= \sum_{i=1}^{s} A_1(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} \cdot k_i + \frac{a_i - 1}{2}\right) \cdot a_i \cdot 10^{k_i} + \sum_{i=2}^{s} a_i \cdot 10^{k_i} \left(\sum_{j=1}^{i-1} a_j\right)$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2}\right) \cdot a_i \cdot 10^{k_i}.$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} d_s^2(n) + \sum_{a_1 \cdot 10^{k_1} \le n < x} d_s^2(n)$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} d_s^2(n + a_1 \cdot 10^{k_1})$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s(n) + a_1)^2$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s^2(n) + 2a_1 \cdot d_s(n) + a_1^2)$$

$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot (x - a_1 \cdot 10^{k_1})$$
$$+ 2a_1 A_1(x - a_1 \cdot 10^{k_1}) + A_2(x - a_1 \cdot 10^{k_1})$$

$$= \cdots\cdots\cdots$$

$$= \sum_{i=1}^{s} A_2(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i^2(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}) + \sum_{i=1}^{s} 2a_i A_1(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \frac{(a_i - 1)(2a_i - 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$
$$+ \sum_{i=2}^{s} a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j^2\right) + \sum_{i=2}^{s} (9k_i + a_i - 1) \cdot a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j\right)$$
$$+ 2 \sum_{i=2}^{s} \left(\sum_{j=1}^{i-1} a_j\right) \cdot a_i \cdot \left(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}\right)$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$
$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j}\right] \cdot \left(\sum_{j=1}^{i-1} a_j\right).$$

This completes the proof of the second part of the Theorem.

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