On a Smarandache Partial Perfect Additive Sequence

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Abstract: The sequence defined through $a_{2k+1}=a_{k+1}-1$, $a_{2k+2}=a_{k+1}+1$ for $k\geq1$ with $a_1=a_2=1$ is studied in detail. It is proved that the sequence is neither convergent nor periodic - questions which have recently been posed. It is shown that the sequence has an amusing oscillating behavior and that there are terms that approach $\pm\infty$ for a certain type of large indices.

Definition of Smarandache perfect f_p sequence: If f_p is a p-ary relation on$\{a_1, a_2, a_3, \ldots\}$ and $f_p(a_i, a_{i+1}, a_{i+2}, \ldots a_{i+p-1})= f_p(a_j, a_{j+1}, a_{j+2}, \ldots a_{j+p-1})$ for all a_i, a_j and all $p>1$, then $\{a_n\}$ is called a Smarandache perfect f_p sequence.

If the defining relation is not satisfied for all a_i,a_j or all p then $\{a_n\}$ may qualify as a Smarandache partial perfect f_p sequence.

The purpose of this note is to answer some questions posed in an article in the Smarandache Notions Journal, vol. 11 [1] on a particular Smarandache partial perfect sequence defined in the following way:

$$a_1=a_2=1$$
$$a_{2k+1}=a_{k+1}-1,\ k\geq1 \quad (1)$$
$$a_{2k+2}=a_{k+1}+1,\ k\geq1 \quad (2)$$

Adding both sides of the defining equations results in $a_{2k+2}+a_{2k+1}=2a_{k+1}$ which gives

$$\sum_{i=1}^{2n} a_i = 2\sum_{i}^{n} a_i \quad (3)$$

Let n be of the form $n=k\cdot2^m$. The summation formula now takes the form

$$\sum_{i=1}^{k\cdot2^m} a_i = 2^m\sum_{i}^{k} a_i \quad (4)$$

From this we note the special cases $\sum_{i=1}^{4} a_i = 4, \sum_{i=1}^{8} a_i = 8, \ldots \sum_{i=1}^{2^m} a_i = 2^m$.

The author of the article under reference poses the questions: "Can you, readers, find a general expression of a_n (as a function of n)? Is the sequence periodical, or convergent or bounded?"

The first 25 terms of this sequence are[1]:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a_k	1	1	0	2	-1	1	1	3	-2	0	0	2	0	2	2	4	-3	-1	-1	1	-1	1	1	3	-1

[1] The sequence as quoted in the article under reference is erroneous as from the thirteenth term.

It may not be possible to find a general expression for a_n in terms of n. For computational purposes, however, it is helpful to unify the two defining equations by introducing the δ-function defined as follows:

$$\delta(n)=\begin{cases} -1 \text{ if } n\equiv 0 \pmod 2 \\ 1 \text{ if } n\equiv 1 \pmod 2 \end{cases} \qquad (5)$$

The definition of the sequence now takes the form:

$$a_1=a_2=1$$

$$a_n = a_{(n+\frac{1+\delta(n)}{2})/2} - \delta(n) \qquad (6)$$

A translation of this algorithm to computer language was used to calculated the first 3000 terms of this sequence. A feeling for how this sequence behaves may be best conveyed by table 1 of the first 136 terms, where the switching between positive, negative and zero terms have been made explicit.

Before looking at some parts of this calculation let us make a few observations.

Although we do not have a general formula for a_n we may extract very interesting information in particular cases. Successive application of (2) to a case where the index is a power of 2 results in:

$$a_{2^m} = a_{2^{m-1}} + 1 = a_{2^{m-2}} + 2 = \ldots = a_2 + m - 1 = m \qquad (7)$$

This simple consideration immediately gives the answer to the main question:

The sequence is neither periodic nor convergent.

We will now consider the difference a_n-a_{n-1} which is calculated using (1) and (2). It is necessary to distinguish between n even and n odd.

1. $n=2k$, $k\geq 2$.

$$a_{2k}-a_{2k-1}=2 \text{ (exception: } a_2-a_1=0) \qquad (8)$$

2. $n=k\cdot 2^m+1$ where k is odd.

$$a_{k\cdot 2^m+1} - a_{k\cdot 2^m} = a_{k\cdot 2^{m-1}+1} - 1 - a_{k\cdot 2^{m-1}} - 1 = \ldots a_{k+1} - a_k - 2m = \begin{cases} 1\text{-}2m \text{ if } k=1 \\ 2\text{-}2m \text{ if } k>1 \end{cases} \qquad (9)$$

In particular

a_{2k+1}-a_{2k}= 0 if $k\geq 3$ is odd.

Table 1. The first 136 terms of the sequence

n	a_n	a_{n+1}	...	etc											
1	1	1													
3	0														
4	2														
5	-1														
6	1	1	3												
9	-2														
10	0	0													
12	2														
13	0														
14	2	2	4												
17	-3	-1	-1												
20	1														
21	-1														
22	1	1	3												
25	-1														
26	1	1	3	1	3	3	5								
33	-4	-2	-2												
36	0														
37	-2														
38	0	0													
40	2														
41	-2														
42	0	0													
44	2														
45	0														
46	2	2	4												
49	-2														
50	0	0													
52	2														
53	0														
54	2	2	4												
57	0														
58	2	2	4	2	4	4	6								
65	-5	-3	-3	-1	-3	-1	-1								
72	1														
73	-3	-1	-1												
76	1														
77	-1														
78	1	1	3												
81	-3	-1	-1												
84	1														
85	-1														
86	1	1	3												
89	-1														
90	1	1	3	1	3	3	5								
97	-3	-1	-1												
100	1														
101	-1														
102	1	1	3												
105	-1														
106	1	1	3	1	3	3	5								
113	-1														
114	1	1	3	1	3	3	5	1	3	3	5	3	5	5	7
129	-6	-4	-4	-2	-4	-2	-2								
136	0														

The big drop. The sequence shows an interesting behaviour around the index 2^m. We have seen that $a_{2^m} = m$. The next term in the sequence calculated from (9) is $m+1-2\cdot m=-m+1$. This makes for the spectacular behaviour shown in diagrams 1 and 2. The sequence gradually struggles to get to a peak for $n=2^m$ where it drops to a low and starts working its way up again. There is a great similarity between the oscillating behaviour shown in the two diagrams. In diagram 3 this behaviour is illustrated as it occurs between two successive peaks.

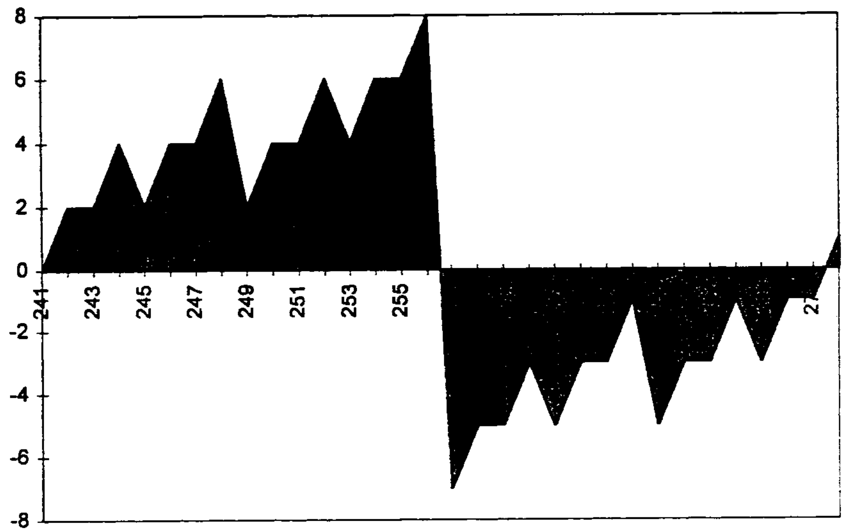

Diagram 1. a_n as a function of n around $n=2^8$ illustrating the "big drop"

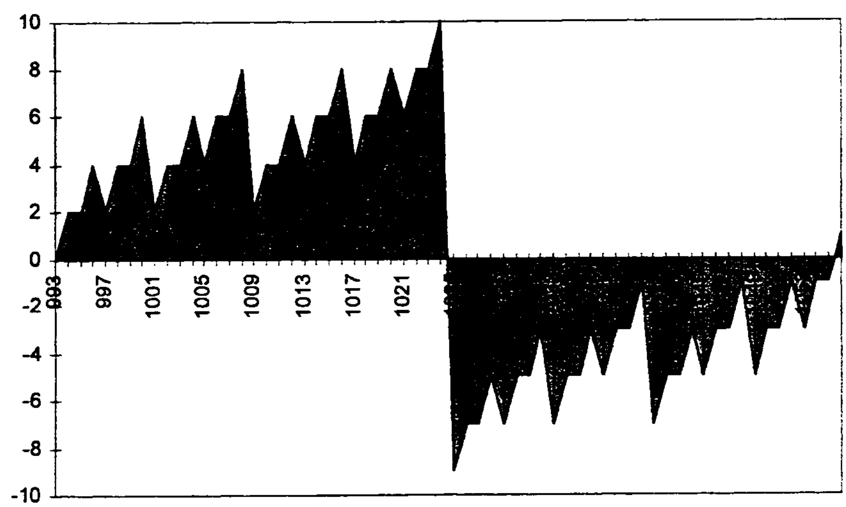

Diagram 2. a_n as a function of n around $n=2^{10}$ illustrating the "big drop"

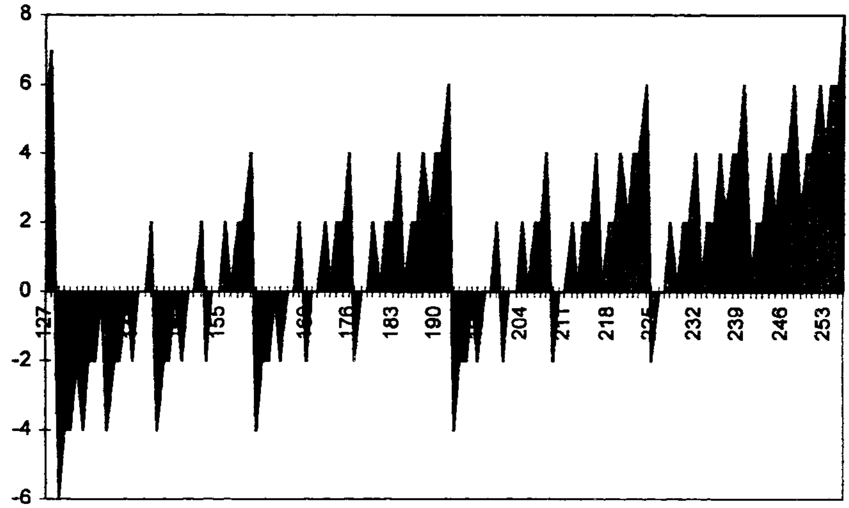

Diagram 3. The oscillating behaviour of the sequence between the peaks for $n=2^7$ and $n=2^8$.

When using the defining equations (1) and (2) to calculate elements of the sequence it is necessary to have in memory the values of the elements as far back as half the current index. We are now in a position to generate preceding and proceeding elements to a given element by using formulas based on (8) and (9).

The forward formulas:

$$a_n = \begin{cases} a_{n-1}+2 & \text{when } n=2k,\ k>1 \\ a_{n-1}+1-2m & \text{when } n=2^m+1 \\ a_{n-1}+2-2m & \text{when } n=k\cdot 2^m+1,\ k>1 \end{cases} \qquad (10)$$

Since we know that $a_{2^m} = m$ it will also prove useful to calculate a_n from a_{n+1}.

The reverse formulas:

$$a_n = \begin{cases} a_{n+1}-2 & \text{when } n=2k-1,\ k>1 \\ a_{n+1}-1+2m & \text{when } n=2^m \\ a_{n+1}-2+2m & \text{when } n=k\cdot 2^m,\ k>1 \end{cases} \qquad (11)$$

Finally let's use these formulas to calculate some terms forwards and backwards from one known value say $a_{4096}=12$ ($4096=2^{12}$). It is seen that a_n starts from 0 at n=4001, makes its big drop to -11 for n=4096 and remains negative until n=4001. For an even power of 2 the mounting sequence only has even values and the descending sequence only odd values. For odd powers of 2 it is the other way round.

Table 2. Values of a_n around $n=2^{12}$.

4095	4094	4093	4092	4091	4090	4089	4088	4087	4086	4085	4084	4083	4082	4081	4080	4079	4078 ...	4001
10	10	8	10	8	8	6	10	8	8	6	8	6	6	4	10	8	8 ...	0
4096	4097	4098	4099	4100	4101	4102	4103	4104	4105	4106	4107	4108	4109	4110	4111	4112	4113 ...	4160
12	-11	-9	-9	-7	-9	-7	-7	-5	-9	-7	-7	-5	-7	-5	-5	-3	-9 ...	1

References:

1. M. Bencze, Smarandache Relationships and Subsequences, *Smarandache Notions Journal. Vol. 11, No 1-2-3, pgs 79-85.*