On certain generalizations of the Smarandache function

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. The famous Smarandache function is defined by $S(n) := \min\{k \in \mathbf{N}^* : n|k!\}$, $n \geq 1$ positive integer. This arithmetical function is connected to the number of divisors of n, and other important number theoretic functions (see e.g. [6], [7], [9], [10]). A very natural generalization is the following one: Let $f : \mathbf{N}^* \to \mathbf{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbf{N}^*$ there exists at least a $k \in \mathbf{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbf{N}^* \to \mathbf{N}^*$ defined by

$$F_f(n) = \min\{k \in \mathbf{N} : n|f(k)\}. \tag{1}$$

Since every subset of natural numbers is well-ordered, the definition (1) is correct, and clearly $F_f(n) \geq 1$ for all $n \in \mathbf{N}^*$.

Examples. 1) Let $id(k) = k$ for all $k \geq 1$. Then clearly (P_1) is satisfied, and

$$F_{id}(n) = n. \tag{2}$$

2) Let $f(k) = k!$. Then $F_!(n) = S(n)$ - the Smarandache function.

3) Let $f(k) = p_k!$, where p_k denotes the kth prime number. Then

$$F_f(n) = \min\{k \in \mathbf{N}^* : \ n | p_k!\}. \tag{3}$$

Here (P_1) is satisfied, as we can take for each $n \geq 1$ the least prime greater than n.

4) Let $f(k) = \varphi(k)$, Euler's totient. First we prove that (P_1) is satisfied. Let $n \geq 1$ be given. By Dirichlet's theorem on arithmetical progressions ([1]) there exists a positive integer a such that $k = an + 1$ is prime (in fact for infinitely many a's). Then clearly $\varphi(k) = an$, which is divisible by n.

We shall denote this function by F_φ. (4)

5) Let $f(k) = \sigma(k)$, the sum of divisors of k. Let k be a prime of the form $an - 1$, where $n \geq 1$ is given. Then clearly $\sigma(n) = an$ divisible by n. Thus (P_1) is satisfied. One obtains the arithmetical function F_σ. (5)

2. Let $A \subset \mathbf{N}^*$, $A \neq \emptyset$ a nonvoid subset of $\mathbf{N}$, having the property:

(P_2) For each $n \geq 1$ there exists $k \in A$ such that $n | k!$.

Then the following arithmetical function may be introduced:

$$S_A(n) = \min\{k \in A : \ n | k!\}. \tag{6}$$

Examples. 1) Let $A = \mathbf{N}^*$. Then $S_{\mathbf{N}}(n) \equiv S(n)$ - the Smarandache function.

2) Let $A = \mathbf{N}_1$ = set of odd positive integers. Then clearly (P_2) is satisfied. (7)

3) Let $A = \mathbf{N}_2$ = set of even positive integers. One obtains a new Smarandache-type function. (8)

4) Let $A = P$ = set of prime numbers. Then $S_P(n) = \min\{k \in P : \ n | k!\}$. We shall

denote this function by $P(n)$, as we will consider more closely this function. (9)

3. Let $g : \mathbf{N}^* \to \mathbf{N}^*$ be a given arithmetical function. Suppose that g satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$. (10)

Let the function $G_g : \mathbf{N}^* \to \mathbf{N}^*$ be defined as follows:

$$G_g(n) = \max\{k \in \mathbf{N}^* : \ g(k)|n\}. \tag{11}$$

This is not a generalization of $S(n)$, but for $g(k) = k!$, in fact one obtains a "dual"-function of $S(n)$, namely

$$G_!(n) = \max\{k \in \mathbf{N}^* : \ k!|n\}. \tag{12}$$

Let us denote this function by $S_*(n)$.

There are many other particular cases, but we stop here, and study in more detail some of the above stated functions.

4. The function $P(n)$

This has been defined in (9) by: the least prime p such that $n|p!$. Some values are: $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11, \ldots$

Proposition 1. For each prime p one has $P(p) = p$, and if n is squarefree, then $P(n) =$ greatest prime divisor of n.

Proof. Since $p|p!$ and $p \nmid q!$ with $q < p$, clearly $P(p) = p$. If $n = p_1p_2 \ldots p_r$ is squarefree, with $p_1, \ldots, p_r$ distinct primes, if $p_r = \max\{p_1, \ldots, p_r\}$, then $p_1 \ldots p_r | p_r!$. On the other hand, $p_1 \ldots p_r \nmid q!$ for $q < p_r$, since $p_r \nmid q!$. Thus p_r is the least prime with the required property.

The calculation of $P(p^2)$ is not so simple but we can state the following result:

Proposition 2. One has the inequality $P(p^2) \geq 2p+1$. If $2p+1 = q$ is prime, then $P(p^2) = q$. More generally, $P(p^m) \geq mp+1$ for all primes p and all integers m. There is equality, if $mp+1$ is prime.

Proof. From $p^2|(1 \cdot 2 \dots p)(p+1)\dots(2p)$ we have $p^2|(2p)!$. Thus $P(p^2) \geq 2p+1$. One has equality, if $2p+1$ is prime. By writing $p^m|\underbrace{1 \cdot 2 \dots p}\underbrace{(p+1)\dots 2p}\dots\underbrace{[(m-1)p+1]\dots mp}$, where each group of p consecutive terms contains a member divisible by p, one obtains $P(p^m) \geq mp+1$.

Remark. If $2p+1$ is not a prime, then clearly $P(p^2) \geq 2p+3$.

It is not known if there exist infinitely many primes p such that $2p+1$ is prime too (see [4]).

Proposition 3. The following double inequality is true:

$$2p+1 \leq P(p^2) \leq 3p-1 \tag{13}$$

$$mp+1 \leq P(p^m) \leq (m+1)p-1 \tag{14}$$

if $p \geq p_0$.

Proof. We use the known fact from the prime number theory ([1], [8]) tha for all $a \geq 2$ there exists at least a prime between $2a$ and $3a$. Thus between $2p$ and $3p$ there is at least a prime, implying $P(p^2) \leq 3p-1$. On the same lines, for sufficiently large p, there is a prime between mp and $(m+1)p$. This gives the inequality (14).

Proposition 4. For all $n, m \geq 1$ one has:

$$S(n) \leq P(n) \leq 2S(n)-1 \tag{15}$$

and

$$P(nm) \leq 2[P(n) + P(m)] - 1 \tag{16}$$

where $S(n)$ is the Smarandache function.

Proof. The left side of (15) is a consequence of definitions of $S(n)$ and $P(n)$, while the right-hand side follows from Chebyshev's theorem on the existence of a prime between a and $2a$ (where $a = S(n)$, when $2a$ is not a prime).

For the right side of (16) we use the inequality $S(mn) \leq S(n) + S(m)$ (see [5]): $P(nm) \leq 2S(nm) - 1 \leq 2[S(n) + S(m)] - 1 \leq 2[P(n) + P(m)] - 1$, by (15).

Corollary.

$$\lim_{n\to\infty} \sqrt[n]{P(n)} = 1. \tag{17}$$

This is an easy consequence of (15) and the fact that $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$. (For other limits, see [6]).

5. The function $S_*(n)$

As we have seen in (12), $S_*(n)$ is in certain sense a dual of $S(n)$, and clearly $(S_*(n))!|n|(S(n))!$ which implies

$$1 \leq S_*(n) \leq S(n) \leq n \tag{18}$$

thus, as a consequence,

$$\lim_{n\to\infty} \sqrt[n]{\frac{S_*(n)}{S(n)}} = 1. \tag{19}$$

On the other hand, from known properties of S it follows that

$$\liminf_{n\to\infty} \frac{S_*(n)}{S(n)} = 0, \quad \limsup_{n\to\infty} \frac{S_*(n)}{S(n)} = 1. \tag{20}$$

For odd values n, we clearly have $S_*(n) = 1$.

Proposition 5. For $n \geq 3$ one has

$$S_*(n! + 2) = 2 \tag{21}$$

and more generally, if p is a prime, then for $n \geq p$ we have

$$S_*(n! + (p-1)!) = p - 1. \tag{22}$$

Proof. (21) is true, since $2|(n!+2)$ and if one assumes that $k!|(n!+2)$ with $k \geq 3$, then $3|(n!+2)$, impossible, since for $n \geq 3$, $3|n!$. So $k \leq 2$, and remains $k = 2$.

For the general case, let us remark that if $n \geq k+1$, then, since $k|(n!+k!)$, we have $S_*(n!+k!) \geq k$.

On the other hand, if for some $s \geq k+1$ we have $s!|(n!+k!)$, by $k+1 \leq n$ we get $(k+1)|(n!+k!)$ yielding $(k+1)|k!$, since $(k+1)|n!$. So, if $(k+1)|k!$ is not true, then we have

$$S_*(n! + k!) = k. \tag{23}$$

Particularly, for $k = p-1$ (p prime) we have $p \nmid (p-1)!$.

Corollary. For infinitely many m one has $S_*(m) = p - 1$, where p is a given prime.

Proposition 6. For all $k, m \geq 1$ we have

$$S_*(k!m) \geq k \tag{24}$$

and for all $a, b \geq 1$,

$$S_*(ab) \geq \max\{S_*(a), S_*(b)\}. \tag{25}$$

Proof. (24) trivially follows from $k!|(k!m)$, while (25) is a consequence of $(S_*(a))!|a \Rightarrow (S_*(a))!|(ab)$ so $S_*(ab) \geq S_*(a)$. This is true if a is replaced by b, so (25) follows.

Proposition 7. $S_*[x(x-1)\dots(x-a+1)] \geq a$ for all $x \geq a$ (x positive integer).(26)

Proof. This is a consequence of the known fact that the product of $\underline{a}$ consecutive integers is divisible by $a!$.

We now investigate certain properties of $S_*(a!b!)$. By (24) or (25) we have $S_*(a!b!) \geq \max\{a, b\}$. If the equation

$$a!b! = c! \tag{27}$$

is solvable, then clearly $S_*(a!b!) = c$. For example, since $3! \cdot 5! = 6!$, we have $S_*(3! \cdot 5!) = 6$. The equation (27) has a trivial solution $c = k!$, $a = k! - 1$, $b = k$. Thus $S_*(k!(k!-1)!) = k$.

In general, the nontrivial solutions of (27) are not known (see e.g. [3], [1]).

We now prove:

Proposition 8. $S_*((2k)!(2k+2)!) = 2k+2$, if $2k+3$ is a prime; (28)

$S_*((2k)!(2k+2)!) \geq 2k+4$, if $2k+3$ is not a prime. (29)

Proof. If $2k+3 = p$ is a prime, (28) is obvious, since $(2k+2)! | (2k)!(2k+2)!$, but $(2k+3)! \nmid (2k)!(2k+2)!$. We shall prove first that if $2k+3$ is not prime, then

$$(2k+3) | (1 \cdot 2 \dots (2k)) \tag{$*$}$$

Indeed, let $2k+3 = ab$, with $a, b \geq 3$ odd numbers. If $a < b$, then $a < k$, and from $2k+3 \geq 3b$ we have $b \leq \frac{2}{3}k + 1 < k$. So $(2k)!$ is divisible by ab, since a, b are distinct numbers between 1 and k. If $a = b$, i.e. $2k+3 = a^2$, then $(*)$ is equivalent with $a^2 | (1 \cdot 2 \dots a)(a+1)\dots(a^2-3)$. We show that there is a positive integer k such that $a+1 < ka \leq a^2 - 3$ or. Indeed, $a(a-3) = a^2 - 3a < a^2 - 3$ for $a > 3$ and $a(a-3) > a+1$ by $a^2 > 4a+1$, valid for $a \geq 5$. For $a = 3$ we can verify $(*)$ directly. Now $(*)$ gives

$$(2k+3)! | (2k)!(2k+2)!, \text{ if } 2k+3 \neq \text{prime} \tag{$**$}$$

implying inequality (29).

For consecutive odd numbers, the product of factorials gives for certain values

$$S_*(3!\cdot 5!) = 6, \quad S_*(5!\cdot 7!) = 8, \quad S_*(7!\cdot 9!) = 10,$$

$$S_*(9!\cdot 11!) = 12, \quad S_*(11!\cdot 13!) = 16, \quad S_*(13!\cdot 15!) = 16, \quad S_*(15!\cdot 17!) = 18,$$

$$S_*(17!\cdot 19!) = 22, \quad S_*(19!\cdot 21!) = 22, \quad S_*(21!\cdot 23!) = 28.$$

The following conjecture arises:

Conjecture. $S_*((2k-1)!(2k+1)!) = q_k - 1$, where q_k is the first prime following $2k+1$.

Corollary. From $(q_k - 1)!|(2k-1)!(2k+1)!$ it follows that $q_k > 2k+1$. On the other hand, by $(2k-1)!(2k+1)!|(4k)!$, we get $q_k \le 4k-3$. Thus between $2k+1$ and $4k+2$ there is at least a prime q_k. This means that the above conjecture, if true, is stronger than Bertrand's postulate (Chebyshev's theorem [1], [8]).

6. Finally, we make some remarks on the functions defined by (4), (5), other functions of this type, and certain other generalizations and analogous functions for further study, related to the Smarandache function.

First, consider the function F_φ of (4), defined by

$$F_\varphi = \min\{k \in \mathbf{N}^* :\ n|\varphi(k)\}.$$

First observe that if $n+1 = \text{prime}$, then $n = \varphi(n+1)$, so $F_\varphi(n) = n+1$. Thus

$$n+1 = \text{prime} \ \Rightarrow\ F_\varphi(n) = n+1. \tag{30}$$

This is somewhat converse to the φ-function property

$$n+1 = \text{prime} \ \Rightarrow\ \varphi(n+1) = n.$$

Proposition 9. Let ϕ_n be the nth cyclotomic polynomial. Then for each $a \geq 2$ (integer) one has

$$F_\varphi(n) \leq \phi_n(a) \text{ for all } n. \tag{31}$$

Proof. The cyclotomic polynomial is the irreducible polynomial of grade $\varphi(n)$ with integer coefficients with the primitive roots of order n as zeros. It is known (see [2]) the following property:

$$n|\varphi(\phi_n(a)) \text{ for all } n \geq 1, \text{ all } a \geq 2. \tag{32}$$

The definition of F_φ gives immediately inequality (31).

Remark. We note that there exist in the literature a number of congruence properties of the function φ. E.g. it is known that $n|\varphi(a^n - 1)$ for all $n \geq 1$, $a \geq 2$. But this is a consequence of (32), since $\phi_n(a)|a^n - 1$, and $u|v \Rightarrow \varphi(u)|\varphi(v)$ implies (known property of φ) what we have stated.

The most famous congruence property of φ is the following

Conjecture. (D.H. Lehmer (see [4])) If $\varphi(n)|(n-1)$, then $n =$ prime.

Another congruence property of φ is contained in Euler's theorem: $m|(a^{\varphi(m)} - 1)$ for $(a, m) = 1$. In fact this implies

$$S_*[a^{\varphi(m!)} - 1] \geq m \text{ for } (a, m!) = 1 \tag{33}$$

and by the same procedure,

$$S_*(\varphi(a^{n!} - 1)] \geq n \text{ for all } n. \tag{34}$$

As a corollary of (34) we can state that

$$\limsup_{k\to\infty} S_*[\varphi(k)] = +\infty. \tag{35}$$

(It is sufficient to take $k = a^{n!} - 1 \to \infty$ as $n \to \infty$).

7. In a completely similar way one can define $F_d(n) = \min\{k : n|d(k)\}$, where $d(k)$ is the number of distinct divisors of k. Since $d(2^{n-1}) = n$, one has

$$F_d(n) \le 2^{n-1}. \tag{36}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be the canonical factorization of the number n. Then Smarandache ([9]) proved that $S(n) = \max\{S(p_1^{\alpha_1}), \dots, S(p_r^{\alpha_r})\}$.

In the analogous way, we may define the functions $S_\varphi(n) = \max\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S_\sigma(n) = \max\{\sigma(p_1^{\alpha_1}), \dots, \sigma(p_r^{\alpha_r})\}$, etc.

But we can define $S_\varphi^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, etc. For an arithmetical function f one can define

$$\Delta_f(n) = l.c.m.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}$$

and

$$\delta_f(n) = g.c.d.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}.$$

For the function $\Delta_\varphi(n)$ the following divisibility property is known (see [8], p.140, Problem 6).

If $(a, n) = 1$, then

$$n|[a^{\Delta_\varphi(n)} - 1]. \tag{37}$$

These functions and many related others may be studied in the near (or further) future.

References

[1] T.M. Apostol, *An introduction to analytic number theory*, Springer Verlag, 1976.

[2] M. Deaconescu and J. Sándor, *Variations on a theme by Hurwitz*, Gazeta Mat. (Perf. Met.) A, **8**(1987), No.4, 186-191.

[3] P. Erdös, *Quelques problèmes de théorie des nombres*, L'Enseignement Math. 1963, pp. 81-135.

[4] R.K. Guy, *Unsolved problems in number theory*, Springer Verlag, Second ed. 1994.

[5] M. Le, *An inequality concerning the Smarandache function*, Smarandache Notions J. **9**(1998), No.1-2, 124-125.

[6] J. Sándor, *On certain new inequalities and limits for the Smarandache function*, Smarandache Notion J. **9**(1998), 63-69.

[7] J. Sándor, *On values of arithmetical functions at factorials*, I (submitted).

[8] H.N. Shapiro, *Introduction to the theory of numbers*, Wiley, 1983.

[9] F. Smarandache, *A function in the number theory*, An. Univ. Timişoara, Ser. Mat., vol. **38**(1980), 79-88.

[10] *Smarandache Notion Journal* (collection).

This paper has appeared in *Notes Num. Theory Discr. Math.* (Bulgaria), 5(1999), 2, 41-51.