# On certain new inequalities and limits for the Smarandache function

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## I. Inequalities

1) If n $\geq$ 4 is an even number, then $S(n) \leq \frac{n}{2}$.

—Indeed, $\frac{n}{2}$ is integer, $\frac{n}{2} > 2$, so in $(\frac{n}{2})! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2}$ we can simplify with 2, so $n \mid (\frac{n}{2})!$.

This simplies clearly that $S(n) \leq \frac{n}{2}$.

2) If $n > 4$ is an even number, then $S(n^2) \leq n$

—By $n! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2} \cdots n$, since we can simplify with 2, for $n > 4$ we get that $n^2 | n!$. This clearly implies the above stated inequality. For factonials, the above inequality can be much improved, namely one has:

3) $S\left((m!)^2\right) \leq 2m$ and more generally, $S\left((m!)^n\right) \leq n \cdot m$ for all positive integers $m$ and $n$.

—First remark that $\frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} =$

$= C_{2m}^m \cdot C_{3m}^m \cdots C_{nm}^m$, where $C_n^k = \binom{n}{k}$ denotes a binomial coefficient. Thus $(m!)^n$ divides $(mn)!$, implying the stated inequality. For $n = 2$ one obtains the first part.

4) Let $n > 1$. Then $S\left((n!)^{(n-1)!}\right) \leq n!$

—We will use the well-known result that the product of $n$ consecutive integers is divisible by $n!$. By $(n!)! = 1 \cdot 2 \cdot 3 \cdots n \cdot \left((n+1)(n+2) \cdots 2n\right) \cdots \left((n-1)!-1\right) \cdots (n-1)!\, n$

each group is divisible by $n!$, and there are $(n-1)!$ groups, so $(n!)^{(n-1)!}$ divides $(n!)!$. This gives the stated inequality.

5) For all $m$ and $n$ one has $[S(m), S(n)] \leq S(m \cdot S(n) \leq [m, n]$. where $[a, b]$ denotes the

$\ell \cdot c \cdot m$ of $a$ and $b$.

—If $m = \prod P_i^{a_i}$, $n = \prod q_j^{b_j}$ are the canonical representations of $m$, resp. $n$, then it is well-known that $S(m) = S(P_i^{a_i})$ and $S(n) = S(q_j^{b_j})$, where $S(P_i^{a_i}) = \max\{S(P_i^{a_i}) : i = 1, \cdots, r\}$; $S(q_j^{b_j}) = \max\{S(q_j^{b_j}) : j = 1, \cdots, h\}$, with $r$ and $h$ the number of prime divisors of $m$, resp. $n$. Then clearly $[S(m), S(n)] \leq S(m) \cdot S(n) \leq P_i^{a_i} \cdot q_j^{b_j} \leq [m, n]$

6) $\left(S(m), S(n)\right) \geq \frac{S(m) \cdot S(n)}{mn} \cdot (m, n)$ for all $m$ and $n$

—Since $\left(S(m), S(m)\right) = \frac{S(m) \cdot S(n)}{[S(m), S(n)]} \geq \frac{S(m) \cdot S(n)}{[m, n]} = \frac{S(m) \cdot S(n)}{nm} \cdot (m, n)$

by 5) and the known formula $[m,n] = \frac{mn}{(m,n)}$.

7) $\frac{\left(S(m), S(n)\right)}{(m, n)} \geq \left(\frac{S(mn)}{mn}\right)^2$ for all $m$ and $n$

—Since $S(mn) \leq m\, S(n)$ and $S(mn) \leq n\, S(m)$ (See [1]), we have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn}$,

and the result follows by 6).

8) We have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn} \leq \frac{1}{(mn)}$

—This follows by 7) and the stronger inequality from 6), namely $S(m)\, S(n) \leq [m\, n] = \frac{mn}{(m, n)}$

Corollary $S(mn) \leq \frac{mn}{\sqrt{mn}}$

9) Max $\left\{S(m), S(n)\right\} \geq \frac{S(mn)}{(mn)}$ for all $m$, $n$; where $(m,n)$ denotes the $g \cdot c \cdot d$ of $m$ and $n$.

—We apply the known result: max $\left\{S(m), S(n)\right\} = S([m, n])$ On the other hand, since $[m, n] \mid m \cdot n$, by Corollary 1 from our paper [1] we get $\frac{S(mn)}{mn} \leq \frac{S([m, n])}{[m, n]}$.

Since $[m, n] = \frac{mn}{(m, n)}$,

The result follows:

Remark. Inequality $g)$ compliments Theorem 3 from [1],

namely that max $\left\{S(m), S(n)\right\} \leq S(mn)$.

10) Let $d(n)$ be the number of divisors of $n$. Then $\frac{S(n!)}{n!} \leq \frac{S\left(n^{d(n)/2}\right)}{n^{d(n)/2}}$

—We will use the known relation $\prod_{k|n} k = n^{d(n)/2}$, where the product is extended over all divisors $k$ of $n$. Since this product divides $\prod_{k \leq n} k = n!$, by Corollary 1 from [1] we can write

$\frac{S(n!)}{n!} \leq \frac{S(\prod_{k/n} k)}{\prod_{k/n} k}$, which gives the desired result.

Remark If $n$ is of the form $m^2$, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many $n$ we have $S(n+1) < S(n)$, but for infinitely many $m$ one has $S(m+1) > S(m)$.

—This is a simple application of 1). Indeed, let $n = p - 1$, where $p \geq 5$ is a prime. Then, by 1) we have $S(n) = S(p-1) \leq \frac{P-1}{2} < p$. Since $p = S(p)$, we have $S(p-1) < S(p)$. Let in the same manner $n = p + 1$. Then, as above, $S(p+1) \leq \frac{P+1}{2} < p = S(p)$.

12) Let $p$ be a prime. Then $S(p!+1) > S(p!)$ and $S(p!-1) > S(p!)$

—Clearly, $S(p!) = p$. Let $p!+1 = \prod q_j^{\partial j}$ be the prime factorization of $p!+1$. Here each $q_j > p$, thus $S(p!+1) = S(q_j^{\partial j})$ (for certain $j$) $\geq S(p^{\partial j}) \geq S(p) = p$. The same proof applies to the case $p! - 1$.

Remark: This offers a new proof for $M$).

13) Let $P_k$ be the $kth$ prime number. Then $S(p_1 p_2 \ldots P_k + 1) > S(p_1 p_2 \cdots P_k)$ and



$S(p_1 p_2 \ldots P_k - 1) > S(p_1 p_2 \cdots P_k)$

—Almost the same proof as in 12) is valid, by remarking that $S(p_1 p_2 \cdots P_k) = P_k$ (since $p_1 < p_2 < \cdots < p_k$).

14) For infinitely many $n$ one has $\left(S(n)^2\right) < S(n-1) \cdot S(n+1)$ and for infinitely many $m$, $\left(S(m)\right)^2 > S(m-1) \cdot S(m+1)$.

—By $S(p+1) < p$ and $S(p-1) < p$ (See the proof in 11) we have

$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p)} < \frac{S(p)}{S(p-1)}$. Thus $\left(S(p)\right)^2 > S(p-1) \cdot S(p+1)$.

On the other hand, by putting $x_n = \frac{S(n+1)}{S(n)}$, we shall see in part II,

that $\limsup_{n\to\infty} x_n = +\infty$. Thus $x_{n-1} < x_n$ for infinitely many $n$, giving

$\left(S(n)\right)^2 < S(n-1) \cdot S(n+1)$.

II. Limits:

1) $\liminf_{n\to\infty} \frac{S(n)}{n} = 0$ and $\limsup_{n\to\infty} \frac{S(n)}{n} = 1$

—Clearly, $\frac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \le 2m$, and $\lim_{m\to\infty} \frac{2m}{2m} = 0$, we have $\lim_{m\to\infty} \frac{S(2^m)}{2^m} = 0$, proving the first part. On the other hand, it is well known that $\frac{S(n)}{n} \le 1$. For $n = p_k$ (the $kth$ prime), one has $\frac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

Remark: With the same proof, we can derive that $\liminf_{n\to\infty} \frac{S(n^r)}{n} = 0$ for all integers $r$.

—As above $S(2^{kr}) \le 2kr$, and $\frac{2kr}{2^k} \to 0$ as $k \to \infty$ ($r$ fixed), which gives the result.

2) $\liminf_{n\to\infty} \frac{S(n+1)}{S(n)} = 0$ and $\limsup_{n\to\infty} \frac{S(n+1)}{S(n)} = +\infty$

—Let $p_r$ denote the $rth$ prime. Since $(p_\Lambda ... p_r, 1) = 1$, Dirichlet's theorem on arithmetical progressions assures the existence of a prime $p$ of the form $p = a \cdot p_\Lambda ... p_r - 1$.

Then $S(p+1) = S(ap_\Lambda \cdots p_r) \le a \cdot S(p_\Lambda \cdots p_r)$ $by$ $S(mn) \le mS(n)$ (see [1])

But $S(p_\Lambda \cdots p_r) = max\left\{p_\Lambda, \cdots, p_r\right\} = p_r$. Thus $\frac{S(p+1)}{S(p)} \le \frac{ap_r}{ap_\Lambda \cdots p_r - 1} \le \frac{p_r}{p_\Lambda \cdots p_r - 1} \to 0$ as $r \to \infty$. This gives the first part.

Let now $p$ be a prime of the form $p = bp_\Lambda \cdots p_r + 1$.

Then $S(p-1) = S(bp_{\Lambda}\cdots p_r) \le b\,S(p_{\Lambda}\cdots p_r) = b \cdot p_r$,

and $\frac{S(p-1)}{S(p)} \le \frac{bp_r}{bp_1\cdots p_r+1} \le \frac{p_r}{p_{\Lambda}\cdots p_r} \to 0$ as $r \to \infty$.

3) $\liminf_{n\to\infty} \left[S(n+1) - S(n)\right] = -\infty$ and $\limsup_{m\to\infty} \left[S(n+1) - S(n)\right] = +\infty$

—We have $S(p+1) - S/p) \le \frac{p+1}{2} - p = \frac{-p+1}{2} \to -\infty$ for an odd prime

$p$ (see 1) and 11)$\Big)$. On the other hand, $S(p) - S(p-1) \ge p - \frac{p-1}{2} = \frac{p+1}{2} \to \infty$

(Here $S(p) = p$), where $p-1$ is odd for $p \ge 5$. This finishes the proof.

4) Let $\sigma(n)$ denotes the sum of divisors of $n$. Then $\liminf_{n\to\infty} \frac{S\big(\sigma(n)\big)}{n} = 0$

—This follows by the argument of 2) for $n = p$. Then $\sigma(\varphi) = p+1$ and $\frac{S(p+1)}{p} \to 0$, where

$\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Enter totient function. Then $\liminf_{n\to\infty} \frac{S\big(\varphi(n)\big)}{n} = 0$

—Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p-1$ and $\frac{S(p-1)}{p} = \frac{S(p-1)}{S(p)} \to 0$,

the assertion is proved. The same result could be obtained by taking $n = z^k$. Then, since

$\varphi(2^k) = 2^{k-1}$, and $\frac{S(2^{k-1})}{2^k} \le \frac{2\cdot(k-1)}{2^k} \to o$ as $k \to \infty$, the assertion follows:

6) $\liminf_{n\to\infty} \frac{S\big(S(n)\big)}{n} = 0$ and $\max_{n\in\aleph} \frac{S\big(S(n)\big)}{n} = 1$.

—Let $n = p!$ ($p$ prime). Then, since $S(p!) = p$ and $S(p) = p$, from $\frac{p}{p!} \to 0$ $(p \to \infty)$

we get the first result. Now, clearly $\frac{S\big(S(n)\big)}{n} \le \frac{S(n)}{n} \le 1$. By letting $n = p$ (prime), clearly

one has $\frac{S\big(S(p)\big)}{p} = 1$, which shows the second relation.

7) $\liminf_{n\to\infty} \frac{\sigma\big(S(n)\big)}{S(n)} = 1$.

—Clearly, $\frac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime), $\frac{\sigma(S(p))}{S(p)} = \frac{p+1}{p} \to 1$ as $p \to \infty$.

8) Let Q(n) denote the greatest prime power divisor of $n$. Then $\liminf_{n\to\infty} \frac{\varphi(S(n))}{\partial(n)} = 0$.

—Let $n = p_1^k \cdots p_r^k$ ($k > 1$, fixed). Then, clearly $\partial(n) = p_r^k$.

By $S(n) = S(p_r^k)$ $\left(\text{since } S(p_r^k) > S(p_i^k) \text{ for } i < k\right)$ and $S(p_r^k) = j \cdot p_r$, with $j \leq k$ (which is known) and by $\varphi(j\, p_k) \leq j \cdot \varphi(p_r) \leq k(p_r - 1)$, we get $\frac{\varphi(S(n))}{\partial(n)} \leq \frac{k\cdot(p_r-1)}{p_r^k} \to 0$ as $r \to \infty$ ($k$ fixed).

9) $\lim_{\substack{m\to\infty \\ m \text{ even}}} \frac{S(m^2)}{m^2} = 0$

—By 2) we have $\frac{S(m^2)}{m^2} \leq \frac{1}{m}$ for $m > 4$, even. This clearly inplies the above remark.

Remark. It is known that $\frac{S(m)}{m} \leq \frac{2}{3}$ if $m \neq 4$ is composite. From $\frac{S(m^2)}{m^2} \leq \frac{1}{m} < \frac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\liminf_{n\to\infty} \frac{\sigma(S(n))}{n} = 0$

—By $\sigma(n) = \sum_{d/n} d = n \sum_{d/n} \frac{1}{d} \leq n \sum_{d=1}^{n} \frac{1}{d} < n \cdot (2 \log n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus $\frac{\sigma(S(n))}{n} < \frac{2\, S(n) \log S(n)}{n}$. For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\frac{4k \log 2k}{2^k} \to 0$ ($k \to \infty$), the result follows.

11) $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$

—This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt[n]{S(n)} \leq \sqrt[n]{n}$; and by $\sqrt[n]{n} \to 1$ as $n \to \infty$. However, 11) is one of a (few) limits, which exists for the Smarandache function.

Finally, we shall prove that:

12) $\limsup_{n\to\infty} \frac{\sigma(n\, S(n))}{n\, S(n)} = +\infty$.

—We will use the facts that $S(p!) = p$, $\frac{\sigma(p!)}{p!} = \sum_{d|p!} \frac{1}{d} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{p} \to \infty$ as $p \to \infty$, and the inequality $\sigma(ab) \geq a\,\sigma(b)$ (see [2] ).

Thus $\frac{\sigma(S(p!)p!}{p! \cdot S(p!)} \geq \frac{S(p!) \cdot \sigma(p!)}{p! \cdot p} = \frac{\sigma(p!)}{p!} \to \infty$. Thus, for the sequence $\{n\} = \{p!\}$, the results follows.


## References

[1] J. Sándor. On certain inequalities involving the Smarandache function. Smarandache Notions J. F (1996), 3 - 6;

[2] J. Sándor. On the composition of some arithmetic functions. Studia Univ. Babes-Bolyai, 34 (1989), F - 14.