ON FOUR PRIME AND COPRIME FUNCTIONS

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Devoted to Prof. Vladimir Shkodrov
for his 70-th birthday

In [1] F. Smarandache discussed the following particular cases of the well-know characteristic functions (see, e.g., [2] or [3]).

1) Prime function: $P : N \to \{0,1\}$, with

$$P(n) = \begin{cases} 0, & \text{if } n \text{ is prime} \\ 1, & \text{otherwise} \end{cases}$$

More generally: $P_k : N^k \to \{0,1\}$, where $k \geq 2$ is an integer, and

$$P_k(n_1, n_2, ..., n_k) = \begin{cases} 0, & \text{if } n_1, n_2, ..., n_k \text{ are all prime numbers} \\ 1, & \text{otherwise} \end{cases}$$

2) Coprime function is defined similarly: $C_k : N^k \to \{0,1\}$, where $k \geq 2$ is an integer, and

$$C_k(n_1, n_2, ..., n_k) = \begin{cases} 0, & \text{if } n_1, n_2, ..., n_k \text{ are coprime numbers} \\ 1, & \text{otherwise} \end{cases}$$

Here we shall formulate and prove four assertions related to these functions.

THEOREM 1: For each $k, n_1, n_2, ..., n_k$ natural numbers:

$$P_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k} (1 - P(n_i)).$$

Proof: Let the given natural numbers $n_1, n_2, ..., n_k$ be prime. Then, by definition

$$P_k(n_1, ..., n_k) = 0.$$

In this case, for each i ($1 \leq i \leq k$):

$$P(n_i) = 0,$$

i.e.,

$$1 - P(n_i) = 1.$$

Therefore

$$\prod_{i=1}^{k} (1 - P(n_i)) = 1,$$

i.e.,

$$1 - \prod_{i=1}^{k} (1 - P(n_i)) = 0 = P_k(n_1, ..., n_k). \tag{1}$$

If at least one of the natural numbers $n_1, n_2, ..., n_k$ is not prime, then, by definition

$$P_k(n_1, ..., n_k) = 1.$$

In this case, there exists at least one i ($1 \leq i \leq k$) for which:

$$P(n_i) = 1,$$

i.e.,

$$1 - P(n_i) = 0.$$

Therefore

$$\prod_{i=1}^{k} (1 - P(n_i)) = 0,$$

i.e.,

$$1 - \prod_{i=1}^{k} (1 - P(n_i)) = 1 = P_k(n_1, ..., n_k). \tag{2}$$

The validity of the theorem follows from (1) and (2).

Similarly it can be proved

THEOREM 2: For each $k, n_1, n_2, ..., n_k$ natural numbers:

$$C_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (1 - C_2(n_i, n_j)).$$

Let $p_1, p_2, p_3, ...$ be the sequence of the prime numbers ($p_1 = 2, p_2 = 3, .p_3 = 5, ...$).

Let $\pi(n)$ be the number of the primes less or equal to n.

THEOREM 3: For each natural number n:

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n) = P(n).$$

Proof: Let n be a prime number. Then

$$P(n) = 0$$

and

$$p_{\pi(n)} = n.$$

Therefore

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n) = C_{\pi(n)}(p_1, p_2, ..., p_{\pi(n)-1}, p_{\pi(n)}) = 0,$$

because the primes $p_1, p_2, ..., p_{\pi(n)-1}, p_{\pi(n)}$ are also coprimes.

Let n be not a prime number. Then

$$P(n) = 1$$

and

$$p_{\pi(n)} < n.$$

Therefore

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n) = C_{\pi(n)+1}(p_1, p_2, ..., p_{\pi(n)-1}, n) = 1,$$

because, if n is a composite number, then it is divited by at least one of the prime numbers $p_1, p_2, ..., p_{\pi(n)-1}$.

With this the theorem is proved.

Analogically, it is proved the following

THEOREM 4: For each natural number n:

$$P(n) = 1 - \prod_{i=1}^{\pi(n)+P(n)-1} (1 - C_2(p_i, n)).$$

COROLLARY: For each natural numbers $k, n_1, n_2, ..., n_k$:

$$P_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k} \prod_{j=1}^{\pi(n_i)+P(n_i)-1} (1 - C_2(p_j, n_i)).$$

These theorems show the connections between the prime and coprime functions. Clearly, it is the C_2 function basing on which all the rest of functions above can be represented.

REFERENCES:

[1] Smarandache, F., Collected Papers, Vol. II, Kishinev University Press, Kishinev, 1997.

[2] Grauert H., Lieb I., Fischer W, Differential- und Integralrechnung, Springer-Verlag, Berlin, 1967.

[3] Yosida K., Functional Analysis, Springer-Verlag, Berlin, 1965.