ON SOME CONVERGENT SERIES

by Emil Burton

<u>*Notations :*</u>

N^* set of integers 1 , 2 , 3 , . . .
$d(n)$ the number of divisors of n .
$S(n)$ the Smarandache function $S : N^* \to N^*$.
$S(n)$ is the smallest integer m with the property that $m!$ is divisible by n
R set of real numbers.

In this article we consider the series $\sum_{k=1}^{\infty} f(S(k))$.

$f : N^* \to R$ is a function which satisfies any conditions.

<u>*Proposition 1.*</u> Let $f : N \to R$ be a function which satisfies condition :

$$f(t) \leq \frac{c}{t^{\alpha}(d(t!)-d((t-1)!))}$$

for every $t \in N^*$, $\alpha > 1$ constant, $c > 0$ constant.

Then the series $\sum_{k=1}^{\infty} f(S(k))$ is convergent.

<u>*Proof:*</u> Let us denote by m_t the number of elements of the set

$M_t = \{ k \in N^* ; S(k) = t \} = \{ k \in N^* ; k \mid t! \text{ and } k \nmid (t-1)! \}$.

It follows that $m_t = d(t!) - d((t-1)!)$.

$$\sum_{k=1}^{\infty} f(S(k)) = \sum_{t=1}^{\infty} m_t f(t)$$

We have $m_t \cdot f(t) \leq m_t \cdot \frac{c}{t^{\alpha} m_t} = \frac{c}{t^{\alpha}}$.

It is well - known that $\sum_{t=1}^{\infty} \frac{1}{t^{\alpha}}$ is convergent if $\alpha > 1$.

Therefore $\sum_{k=1}^{\infty} f(S(k)) < \infty$.

It is known that $d(n) < 2\sqrt{n}$ if $n \in N^*$ (2)
and it is obvious that $m_t < d(t!)$ (3)
We can show that

$$\sum_{k=1}^{\infty} \left(S(k)^p \sqrt{S(k)!} \right)^{-1} < \infty \text{ , } p > 1 \qquad (4)$$

$$\sum_{k=1}^{\infty} \left(S(k)! \right)^{-1} < \infty \qquad (5)$$

$$\sum_{k=1}^{\infty} (S(1)\,!\, S(2)\,! \ldots S(k)\,!)^{-1/k} < \infty \qquad (6)$$

$$\sum_{k=2}^{\infty} (S(k) \sqrt{S(k)!} (\log S(k))^p)^{-1} < \infty , \quad p > 1 \qquad (7)$$

Write $f(S(k)) = (S(k)^p \cdot \sqrt{S(k)!}\,)^{-1}$, $f(t) = (t^p \cdot \sqrt{t!}\,)^{-1} = 2(t^p \cdot \sqrt{t!}\,)^{-1} <$
$< 2(t^p \cdot d(t!))^{-1} < 2(t^p \cdot (d(t!) - d((t-1)!)))^{-1}$.

Now use the proposition 1 to get (4).

The convergence of (5) follows from inequality $t\sqrt{t!} < t!$ if $p \in R$, $p > 1$, $t > t_* = [e^{2p+1}]$, $t \in N^*$. Here $[e^{2p+1}]$ means the greatest integer $\leq e^{2p+1}$. The details are left to the reader. To show (6) we can use the Carleman's Inequality : Let $(x_n)_{n \in N^*}$ be a sequence of positive real numbers and $x_n \neq 0$ for some n. Then

$$\sum_{k=1}^{\infty} (x_1 x_2 \cdots x_k)^{1/k} < e \sum_{k=1}^{\infty} x_k \qquad (8)$$

Write $x_k = (S(k)\,!)^{-1}$ and use (8) and (5) to get (6).

It is well-known that

$$\sum_{n=2}^{\infty} (n(\log n)^p)^{-1} < \infty \quad \underline{\text{if and only if}} \quad p > 1 , \; p \in R . \qquad (9)$$

Write $f(t) = (t\sqrt{t!}\,(\log t)^p)^{-1}$, $t \geq 2$, $t \in N^*$. We have

$$\sum_{k=2}^{\infty} (S(k)\sqrt{S(k)\,!}\,(\log S(k))^p)^{-1} = \sum_{k=2}^{\infty} m_t f(t) .$$

$m_t f(t) < d(t\,!)\, f(t) < 2\sqrt{t\,!}\,(t\sqrt{t\,!}\,(\log t)^p)^{-1} = 2\,(t\,(\log t)^p)^{-1}$.

Now use (9) to get (7).

__Remark 1.__ Apply (5) and Cauchy's Condensation Test to see that

$\sum_{k=0}^{\infty} 2^k (S(2^k)\,!)^{-1} < \infty$. This implies that $\lim_{k \to \infty} 2^k (S(2^k)\,!)^{-1} = 0$.

A problem : Test the convergence behaviour of the series

$$\sum_{n=1}^{\infty} (S(k)^p \sqrt{(S(k)-1)\,!}\,)^{-1} . \qquad (10)$$

__Remark 2.__ This problem is more powerful than (4).

Let p_n denote the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, ...).

It is known that $\sum_{n=1}^{\infty} 1/p_n = \infty$. (11)

We next make use of (11) to obtain the following result :

$$\sum_{n=1}^{\infty} S(n)/n^2 = \infty . \qquad (12)$$

We have $\sum_{n=1}^{\infty} S(n)/n^2 \geq \sum_{k=1}^{\infty} S(p_k)/p_k^2 = \sum_{k=1}^{\infty} p_k/p_k^2 = \sum_{k=1}^{\infty} 1/p_k$ (13)

Now apply (13) and (11) to get (12).

We can also show that

$$\sum_{n=1}^{\infty} S(n)/n^{1+p} < \infty \text{ if } p > 1, p \in R. \qquad (14)$$

Indeed, $\sum_{n=1}^{\infty} S(n)/n^{1+p} \leq \sum_{n=1}^{\infty} n/n^{1+p} = \sum_{n=1}^{\infty} 1/n^p < \infty$.

If $0 \leq p \leq 2$, we have $S(n)/n^p \geq S(n)/n^2$.

Therefore $\sum_{n=1}^{\infty} S(n)/n^p = \infty$ if $0 \leq p \leq 2$.

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