ON SOME DIOPHANTINE EQUATIONS

by

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Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by n (Smarandache Function). We shall assume that $S : N^* \to N^*$, $S(1) = 1$. Our purpose is to study a variety of Diophantine equations involving the Smarandache function. We shall determine all solutions of the equations (1), (3) and (8).

(1) $x^{S(x)} = S(x)^x$

(2) $x^{S(y)} = S(y)^x$

(3) $x^{S(x)} + S(x) = S(x)^x + x$

(4) $x^{S(y)} + S(y) = S(y)^x + x$

(5) $S(x)^x + x^2 = x^{S(x)} + S(x)^2$

(6) $S(y)^x + x^2 = x^{S(y)} + S(y)^2$

(7) $S(x)^x + x^3 = x^{S(x)} + S(x)^3$

(8) $S(y)^x + x^3 = x^{S(y)} + S(y)^3$.

For example, let us solve equation (1):

We observe that if $x = S(x)$, then (1) holds.

But $x = S(x)$ if and only if $x \in \{1, 2, 3, 4, 5, 7, ..\} = \{x \in N^*;\ x\text{-prime}\} \cup \{1, 4\}$.

If $x \geq 6$ is not a prime integer, then $S(x) < x$. We can write $x = S(x) + t$, $t \in N^*$, which implies that $S(x)^{S(x)+t} = (S(x) + t)^{S(x)}$. Thus we have $S(x)^t = (1 + \frac{t}{S(t)})^{S(x)}$.

Applying the well-known result $(1 + \frac{k}{n})^n < 3^k$, for $n, k \in N^*$, we have $S(x)^t < 3^t$ which implies that $S(x) < 3$ and consequently $x < 3$. This contradicts our choice of x.

Thus, the solutions of (1) are $A_1 = \{x \in N^*;\ x = \text{prime}\} \cup \{1, 4\}$.

Let us denote by A_k the set of all solutions of the equation (k).

To find A_8 for example, we see that $(S(n), n) \in A_8$ for $n \in N^*$.

Now suppose that $x \neq S(y)$. We can show that (x, y) does not belongs to A_8 as follows: $1 < S(y) < x \Rightarrow S(y) \geq 2$ and $x \geq 3$. On the other hand,

$S(y)^x - x^{S(y)} > S(y)^x - x^x = (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + \ldots + x^{x-1}) \geq$

$(S(y) - x)(S(y)^2 + xS(y) + x^2) = S(y)^3 - x^3$.

Thus, $A_8 = \{(x, y);\ y = n,\ x = S(n),\ n \in N^*\}$.

To find A_3, we se that $x = 1$ implies $S(x) = 1$ and (3) holds.

If $S(x) = x$, (3) also holds.

If $x \geq 6$ is not a prime number, then $x > S(x)$.

Write $x = S(x) + t$, $t \in N^* = \{1, 2, 3, \ldots\}$.

Combining this with (3) yields

$S(x)^{S(x)+t} + S(x) + t = (S(x) + t)^{S(x)} + S(x) \Leftrightarrow S(x)^t + t/S(x)^{S(x)} = (1 + t/S(x))^{S(x)} < 3^t$

which implies $S(x) < 3$. This contradicts our choice of x.

Thus $A_3 = \{ x \in N^* ; x = \text{prime} \} \cup \{ 1 , 4 \}$.
Now, we suppose that the reader is able to find $A_2, A_4, \ldots, A_7$.
We next determine all positive integers x such that $x = \sum_{k^2 < x} k^2$

Write $\quad 1^2 + 2^2 + \ldots + s^2 = x \qquad (1)$

$$s^2 < x \qquad (2)$$

$$(s+1)^2 \geq x \qquad (3)$$

(1) implies $x = s(s+1)(2s+1)/6$. Combining this with (2) and (3) we have $6s^2 < s(s+1)(2s+1)$ and $6(s+1)^2 \geq s(s+1)(2s+1)$. This implies that $s \in \{2, 3\}$.
$s = 2 \Rightarrow x = 5$ and $s = 3 \Rightarrow x = 14$.
Thus $x \in \{ 5, 14 \}$.
In a similar way we can solve the equation $x = \sum_{k^3 < x} k^3$

We find $x \in \{ 9, 36, 100 \}$.
We next show that the set $M_p = \{ n \in N^* ; n = \sum_{k^p < n} k^p , p \geq 2 \}$ has at least $[p/\ln 2] - 2$ elements.

Let $m \in N^*$ such that $m - 1 < p/\ln 2 \qquad (4)$
and $p/\ln 2 < m \qquad (5)$
Write (4) and (5) as:
$2 < e^{p/m-1} \qquad (6)$
$e^{p/m} < 2 \qquad (7)$
Write $x_k = (1 + 1/k)^k$, $y_k = (1 + 1/k)^{k+1}$.
It is known that $x_s < e < y_t$ for every $s, t \in N^*$.
Combining this with (6) and (7) we have
$x_s^{p/m} < e^{p/m} < 2 < e^{p/m-1} < y_t^{p/m-1}$ for every $s, t \in N^*$.
We have $2 < y_t^{p/m-1} = ((t+1)/t)^{(t+1)p/m-1} \leq ((t+1)/t)^p$ if $(t+1)/(m-1) \leq 1$.
So, if $t \leq m - 2$ we have $2 < ((t+1)/t)^p \Leftrightarrow 2\, t^p < (t+1)^p \Leftrightarrow (t+1)^p - t^p > t^p$ (8).
Let $A_p(s)$ denote the sum $1^p + 2^p + \ldots\ldots + s^p$.
Proposition 1. $(t+1)^p > A_p(t)$ for every $t \leq m-2$, $t \in N^*$.
Proof. Suppose that $A_p(t) \geq (t+1)^p \Leftrightarrow A_p(t-1) > (t+1)^p - t^p > t^p \Leftrightarrow$
$A_p(t-2) > t^p - (t-1)^p > (t-1)^p \Leftrightarrow \ldots \Leftrightarrow A_p(1) > 2^p$ which is not true.
It is obvious that $A_p(t) > t^p$ if $t \in N^*$, $2 \leq t \leq m-2$ which implies $A_p(t) \in M_p$ for every $t \in N^*$ and $2 \leq t \leq m-2$.
Therefore card $M_p > m-3 = (m-1) - 2 = [p/\ln 2] - 2$.

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