ON THE 20-th AND THE 21-st SMARANDACHE'S PROBLEMS

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The 20-nd problem from [1] is the following (see also Problem 25 from [2]):

Smarandache divisor products:

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19, 8000, 441, 484, 23,$$
$$331776, 125, 676, 729, 21952, 29, 810000, 31, 32768, 1089, 1156, 1225, 10077696, 37, 1444,$$
$$1521, 2560000, 41, \ldots$$

($P_d(n)$ is the product of all positive divisors of n.)

The 21-st problem from [1] is the following (see also Problem 26 from [2]):

Smarandache proper divisor products:

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1, 13824, 5, 26, 27,$$
$$784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39, 64000, 1, \ldots$$

($p_d(n)$ is the product of all positive divisors of n but n.)

These problems their solutions are well-known and by this reason we shall give more unstandard solutions (see, e.g. [3]).

Let

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

where $p_1 < p_2 < \ldots < p_k$ are different prime numbers and $k, a_1, a_2, \ldots, a_k \geq 1$ are natural numbers. Then

$$P_d(n) = \prod_{d/n} d.$$

Therefore, every divisor of n will be a natural number with the form

$$d = \prod_{i=1}^{k} p_i^{b_i},$$

where $b_1, b_2, \ldots, b_k$ are natural numbers and for every i ($1 \leq i \leq k$): $0 \leq b_i \leq a_i$, i.e.,

$$P_d(n) = \prod_{i=1}^{k} p_i^{c_i},$$

where $c_1, c_2, ..., c_k$ are natural numbers and below we shall discuss their form.

First, we shall note that for fixed where $k, a_1, a_2, ..., a_k,\ p_1, p_2, ..., p_k$ the number of the different divisors of n will be

$$\tau(n) = \prod_{i=1}^{k} (a_i + 1).$$

THEOREM: For every natural number $n = \prod_{i=1}^{k} p_i^{a_i}$:

$$P_d(n) = \prod_{i=1}^{k} p_1^{a_1+1}. \dots .p_{i-1}^{a_{i-1}+1}.p_i^{t_{a_i}}.p_{i+1}^{a_{i+1}+1}. \dots .p_k^{a_k+1}, \tag{1}$$

where $t_q = \frac{q.(q+1)}{2}$ is the $q-$th triangular number.

Proof: When n is a prime number, i.e., $k = a_1 = 1$, the validity of (1) is obvious. Let us assume that (1) is valid for some natural number $m = \sum_{i=1}^{k} a_i$. We shall prove (8) for $m + 1$, i.e., for the natural number $n' = n.p$, where p is a prime number. There are two cases for p.

Case 1: $p \notin \{p_1, p_2, ..., p_k\}$. Then

$$P_d(n') = P_d(n.p) = (P_d(n)).(P_d(n).p^{(a_1+1). \dots .(a_k+1)})$$

(because the first term contains all multipliers of n multiplied by 1 and in the second term - multiplied by p)

$$= (P_d(n))^2.p^{(a_1+1). \dots .(a_k+1)} = (P_d(n))^{a_{k+1}+1}.p^{(a_1+1). \dots .(a_k+1).t_{a_{k+1}}}$$

$$= \prod_{i=1}^{k+1} p_1^{a_1+1}. \dots .p_{i-1}^{a_{i-1}+1}.p_i^{t_{a_i}}.p_{i+1}^{a_{i+1}+1}. \dots .p_{k+1}^{a_{k+1}+1}.$$

Case 2: $p = p_s \in \{p_1, p_2, ..., p_k\}$. Then $n = m.p_s^{a_s}$ and

$$P_d(n') = P_d(n.p) = P(m.p_s^{a_s+1}) = (P_d(m).1).(P_d(m).p_s^{(a_1+1). \dots .(a_{s-1}+1).(a_{s+1}+1). \dots .(a_k+1)})$$

$$.(P_d(m).p_s^{2.(a_1+1). \dots .(a_{s-1}+1).(a_{s+1}+1). \dots .(a_k+1)})$$

$$. \dots .(P_d(m).p_s^{(a_s+1).(a_1+1). \dots .(a_{s-1}+1).(a_{s+1}+1). \dots .(a_k+1)})$$

$$= (P_d(m))^{a_s+1}.p_s^{(a_1+1). \dots .(a_{s-1}+1).(a_{s+1}+1).\dots.(a_k+1).(1+2+\dots+(a_s+1))}$$

$$= (P_d(m))^{a_s+1}.p_s^{(a_1+1). \dots .(a_{s-1}+1).t_{a_s+1}}$$

$$= \prod_{i=1}^{k+1} p_1^{a_1+1}. \dots .p_{i-1}^{a_{i-1}+1}.p_i^{t_{a_i}}.p_{i+1}^{a_{i+1}+1}. \dots .p_{k+1}^{a_{k+1}+1}.$$

Therefore, (1) is valid, i.e., Problem 20 is solved. Using it we can see easily, that

$$P_d(n) = \prod_{i=1}^{k} p_1^{a_1+1}. \dots .p_{i-1}^{a_{i-1}+1}.p_i^{\frac{a_i.(a_i+1)}{2}}.p_{i+1}^{a_{i+1}+1}. \dots .p_k^{a_k+1}$$

$$= \prod_{i=1}^{k} p_i^{\frac{1}{2}.(a_1+1). \dots .(a_k+1)}.p_i^{a_1. \dots .a_k}$$

$$= \prod_{i=1}^{k} n^{\frac{1}{2}.\tau(n)}.n = n.\sqrt{\prod_{i=1}^{k} n^{\tau(n)}},$$

i.e.,

$$P_d(n) = n.\sqrt{\prod_{i=1}^{k} n^{\tau(n)}}, \qquad (2)$$

which is the standard form of the representation of $P_d(n)$.

From (2), having in mind that

$$p_d(n) = \frac{P_d(n)}{n}$$

it is seen directly that the solurion of 21-st problem is

$$p_d(n) = \sqrt{\prod_{i=1}^{k} n^{\tau(n)}},$$

or in the form of (1):

$$p_d(n) = \prod_{i=1}^{k} p_1^{a_1+1}. \dots .p_{i-1}^{a_{i-1}+1}.p_i^{\frac{a_i}{2}}.p_{i+1}^{a_{i+1}+1}. \dots .p_k^{a_k+1}.$$

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[2] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.

[3] T. Nagell, Introduction to Number Theory. John Wiley & Sons, Inc., New York, 1950.