ON THE 45-TH SMARANDACHE'S PROBLEM*

GAO JING AND LIU HUANING

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

ABSTRACT. For any positive integer n, let $k(n)$ be the smallest integer such that $nk(n)$ is a factorial number. In this paper, we study the hybrid mean value of $k(n)$ and the Mangoldt function, and give a sharp asymptotic formula.

1. INTRODUCTION AND RESULTS

For any positive integer n, let $k(n)$ be the smallest integer such that $nk(n)$ is a factorial number. For example, $k(1) = 1$, $k(2) = 1$, $k(3) = 2$, $k(4) = 6$, $k(5) = 24$, $k(6) = 1$, $k(7) = 720$, $\cdots$. Professor F. Smarandache [1] asks us to study the sequence. About this problem, we know very little. The problem is interesting because it can help us to calculate the Smarandache function.

For any prime number p and positive integer n, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by p^n. Professor F. Smarandache [1] also asks us to study this sequence. It seems that $k(n)$ relates to $S_p(n)$. In fact, let $n = p^\alpha$, then we have $k(p^\alpha) = S_p(\alpha)!/p^\alpha$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \cdots, p_r$ are distinct prime numbers. It is not hard to show that

$$k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = \text{Max}\{S_{p_i}(\alpha_i)! | i = 1, 2, \cdots, r\}/(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}).$$

In this paper, we study the hybrid mean value of $k(n)$ and the Mangoldt function, and give a sharp asymptotic formula. That is, we shall prove the following theorems.

Theorem 1. *If $x \geq 2$, we have*

$$\sum_{n < x} \Lambda_1(n) \log k(n) = \frac{1}{2} x^2 \log x + O(x^2),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \text{if } n \text{ is a prime } p\ ; \\ 0, & \text{otherwise.} \end{cases}$$

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Theorem 2. *If $x \geq 2$, we have*

$$\sum_{n \leq x} \Lambda(n) \log k(n) = \frac{1}{2} x^2 \log x + O(x^2),$$

where $\Lambda(n)$ is the Mangoldt function.

It is an unsolved problem whether there exists an asymptotic formula for $\sum_{n \leq x} \log k(n)$. We conjecture that

$$\sum_{n \leq x} \log k(n) = \frac{1}{2} x^2 + O\left(\frac{x^2}{\log x}\right).$$

2. Some Lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. *If $x \geq 2$ we have*

$$\log[x]! = x \log x - x + O(\log x),$$

where $[y]$ denotes the largest integer not exceeding y.

Proof. This is Theorem 3.15 of [2].

Lemma 2. *For any prime number p and positive integer n, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by p^n. Then we have*

$$n(p-1) \leq S_p(n) \leq np.$$

Proof. It is obvious that $S_p(n) \leq np$.

On the other hand, by Theorem 3.14 of [2] we have

$$S_p(n)! = \prod_{p_1 \leq S_p(n)} p_1^{\alpha(p_1)}, \quad \alpha(p_1) = \sum_{m=1}^{\infty} \left[\frac{S_p(n)}{p_1^m}\right],$$

where $\prod_{p_1 \leq x}$ denotes the product over prime numbers not exceeding x. Note that $p^n \mid S_p(n)$, we get

$$n \leq \alpha(p) = \sum_{m=1}^{\infty} \left[\frac{S_p(n)}{p^m}\right] < \sum_{m=1}^{\infty} \frac{S_p(n)}{p^m} = \frac{S_p(n)}{p-1}.$$

This proves Lemma 2.

3. Proofs of the Theorems

In this section, we complete the proofs of the theorems. From Lemma 1 and the definition of $k(n)$ we have

$$\sum_{n\le x}\Lambda_1(n)\log k(n)=\sum_{p\le x}\log p\log(p-1)!$$
$$=\sum_{p<x}\log p\left[(p-1)\log(p-1)-(p-1)+O(\log(p-1))\right]=\sum_{p<x}\left[p\log^2 p+O\left(p\log p\right)\right].$$

Let

$$a(n)=\begin{cases}1, & \text{if } n \text{ is prime;}\\ 0, & \text{otherwise,}\end{cases}$$

then

$$\sum_{n\le x}a(n)=\pi(x)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right).$$

By Abel's identity we have

$$\sum_{p<x}p\log^2 p=\sum_{n<x}a(n)n\log^2 n=\pi(x)\cdot x\log^2 x-\int_2^x\pi(t)\left(\log^2 t+2\log t\right)dt$$
$$=x^2\log x+O(x^2)-\int_2^x\left(t\log t+O(t)\right)dt$$

We can easily get

$$\int_2^x t\log t dt=\frac{1}{2}x^2\log x+O(x^2),$$

Therefore

$$\sum_{p\le x}p\log^2 p=\frac{1}{2}x^2\log x+O(x^2).$$

Similarly we can get

$$\sum_{p<x}p\log p\ll x^2.$$

So we have

$$\sum_{n<x}\Lambda_1(n)\log k(n)=\frac{1}{2}x^2\log x+O(x^2).$$

This proves Theorem 1.

From Lemma 1, Lemma 2 and the definition of $k(n)$ we have

$$\sum_{n\le x}\Lambda(n)\log k(n)=\sum_{p^\alpha<x}\log p\log\left(S_p(\alpha)!/p^\alpha\right)$$
$$=\sum_{p^\alpha<x}\log p\left[S_p(\alpha)\log S_p(\alpha)-S_p(\alpha)+O\left(\log S_p(\alpha)\right)-\alpha\log p\right]$$
$$=\sum_{p^\alpha<x}\left[\alpha p\log^2 p+O(\alpha p\log p\log\alpha)\right]=\sum_{\alpha<\log_2 x}\ \sum_{p\le x^{1/\alpha}}\left[\alpha p\log^2 p+O(\alpha p\log p\log\alpha)\right].$$

Note that

$$\sum_{\alpha<\log_2 x} \sum_{p<x^{1/\alpha}} \alpha p \log^2 p - \sum_{p\leq x} p \log^2 p = \sum_{2\leq\alpha<\log_2 x} \sum_{p\leq x^{1/\alpha}} \alpha p \log^2 p$$

$$\ll \sum_{2\leq\alpha\leq\log_2 x} \alpha x^{2/\alpha} \log^2 x^{1/\alpha} \ll x \log^4 x$$

and

$$\sum_{\alpha\leq\log_2 x} \sum_{p\leq x^{1/\alpha}} \alpha p \log p \log \alpha = \sum_{2\leq\alpha\leq\log_2 x} \sum_{p\leq x^{1/\alpha}} \alpha p \log p \log \alpha$$

$$\ll \sum_{2\leq\alpha\leq\log_2 x} \alpha \log \alpha x^{2/\alpha} \log x^{1/\alpha} \ll x \log^3 x \log\log x,$$

so we have

$$\sum_{n\leq x} \Lambda(n) \log k(n) = \frac{1}{2} x^2 \log x + O(x^2).$$

This completes the proof of Theorem 2.

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References

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