ON THE 49-TH SMARANDACHE'S PROBLEM*

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ABSTRACT. For any prime number p and positive integer n, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by p^n. In this paper, we study the mean value of the Dirichlet series with coefficients $S_p(n)$. We also show that $S_p(n)$ closely relates to Riemann Zeta function, and give a few asymptotic formulae involving $S_p(n)$ and other arithmetic functions.

1. INTRODUCTION AND RESULTS

For any prime number p and positive integer n, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $S_3(5) = 12$, $S_3(6) = 15$, $S_3(7) = 18$, $\cdots$. It is obvious that $p \mid S_p(n)$ and $S_p(n) \leq np$. Professor F. Smarandache [1] asks us to study the sequence. About this problem, we know very little. The problem is interesting because it can help us to calculate the Smarandache function.

It seems that $S_p(n)$ closely relates to Riemann Zeta function. In fact, for real $s > 1$, we consider the Dirichlet series with coefficients $S_p(n)$. The series $\sum S_p(n)n^{-s}$ converges absolutely as $s > 2$ since $S_p(n) \leq np$. In this paper, we study the mean value of the Dirichlet series with coefficients $S_p(n)$, and give a few asymptotic formulae involving $S_p(n)$ and other arithmetic functions.

Theorem 1. *For any given s, we have*

$$\sum_{n=1}^{\infty} \frac{S_p(n)}{n^s} = (p-1)\zeta(s-1) + R_1(s,p), \quad s > 2;$$
$$\sum_{n=1}^{\infty} \frac{\phi(n)S_p(n)}{n^s} = \frac{(p-1)\zeta(s-2)}{\zeta(s-1)} + R_2(s,p), \quad s > 3,$$

where

$$R_1(s,p) \leq \frac{p-1}{\log 2} \sum_{n=1}^{\infty} \frac{\log n + \log p}{n^s}, \quad R_2(s,p) \leq \frac{p-1}{\log 2} \sum_{n=1}^{\infty} \frac{\phi(n)(\log n + \log p)}{n^s}.$$

From our theorem we know that $S_p(n)$ closely relates to Riemann Zeta function. Using our formulae we can calculate the mean value of $S_p(n)$.

Key words and phrases. Primitive numbers; Dirichlet series; Mean value; Asymptotic formula.
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2. Some Lemmas

To complete the proof of the theorem, we need the following lemma.

Lemma 1. *For any prime number p and positive integer n, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by p^n. Then we have*

$$n(p-1) \le S_p(n) \le \left(n + \frac{\log(np)}{\log 2}\right)(p-1).$$

Proof. By Theorem 3.14 of [2] we have

$$S_p(n)! = \prod_{p_1 \le S_p(n)} p_1^{\alpha(p_1)}, \quad \alpha(p_1) = \sum_{m=1}^{\infty} \left[\frac{S_p(n)}{p_1^m}\right],$$

where $\prod_{p_1 \le x}$ denotes the product over prime numbers not exceeding x. Note that $p^n \mid S_p(n)$, we get

$$n \le \alpha(p) = \sum_{m=1}^{\infty} \left[\frac{S_p(n)}{p^m}\right] \le \sum_{m=1}^{\infty} \frac{S_p(n)}{p^m} = \frac{S_p(n)}{p-1}.$$

On the other hand, $p^n \dagger (S_p(n)-1)!$ since $p \mid S_p(n)$. Therefore

$$n-1 \ge \sum_{m=1}^{\infty} \left[\frac{S_p(n)-1}{p^m}\right] \ge \sum_{m=1}^{\infty} \frac{S_p(n)-1}{p^m} - \sum_{\substack{m=1 \\ p^m \le S_p(n)-1}}^{\infty} 1 \ge \frac{S_p(n)-1}{p-1} - \frac{\log(np)}{\log 2}.$$

So we have

$$S_p(n) \le \left(n - 1 + \frac{\log(np)}{\log 2}\right)(p-1) + 1 \le \left(n + \frac{\log(np)}{\log 2}\right)(p-1).$$

This proves Lemma 1.

3. Proof of the Theorem

In this section, we complete the proof of Theorem 1. From Lemma 1 we have

$$S_p(n) = n(p-1) + O\left((p-1)(\log n + \log p)\right).$$

From Theorem 3.2 of [2] we immediately get

$$\sum_{n=1}^{\infty} \frac{S_p(n)}{n^s} = (p-1)\zeta(s-1) + R_1(s,p), \quad s > 2,$$

where

$$R_1(s,p) \le \frac{p-1}{\log 2} \sum_{n=1}^{\infty} \frac{\log n + \log p}{n^s}.$$

Similarly we can deduce other formula.

This completes the proof of Theorem 1.

References

1. F. Smarandache, *Only problems, not solutions*, Xiquan Publ. House, Chicago, 1993.
2. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.