On the 80th Problem of F.Smarandache(I)

He Xiaolin Guo Jinbao
College of Mathematics and Computer Science,Yanan University,Shaanxi China 716000

Abstract Using analytic methold,this paper studies the first power mean of $a(n)$ and its generation, and gives a mean value formula,where $a(n)$ is the sequence in problem 80 of" only problems not solutions" which was presented by professor F.Smarandache.

Keywords number-theoretic function; mean-value; asymptotic formula

In 1993,number-theoretic expert F.Smarandache presented 100 unsolved problems in [1],it arose great interests for scholars.Among them,the 80th problem is:

Square root: 0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6, 6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8,8,$\cdots$

Study this sequences.

We donote the sequence in problem 80 as $a(n)$,it is not difficult to show that $a(n) = [\sqrt{n}\,]$,where $[x]$ is the maximal integer that is no more than x.

1. Mean-value about $a(n)$

Theorem 1 Let n be a positive integer , and $a(n) = [\sqrt{n}\,]$,then

$$\sum_{n\le x} a(n) = \sum_{n\le x}[\sqrt{n}\,] = \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x + O(x^{\frac{1}{2}})$$

Proof For an arbitray positive NUMBER x,there must existS positive integer N,such that $N^2 \le x < (N+1)^2$,so we have

$$\begin{aligned}
&\sum_{n\le x} a(n) = \sum_{n\le x}[\sqrt{n}\,] \\
&= \sum_{1^2\le i<2^2}[\sqrt{i}\,] + \sum_{2^2\le i<3^2}[\sqrt{i}\,] + \cdots + \sum_{N^2\le i\le x<(N+1)^2}[\sqrt{i}\,] + O(N) \\
&= 3\cdot 1 + 5\cdot 1 + \cdots + [(N+1)^2 - N^2]\cdot N + O(N) \\
&= \sum_{j\le N}(2j+1)j + O(N) \\
&= 2\sum_{j\le N} j^2 + \sum_{j\le N} j + O(N) \\
&= 2\cdot\frac{1}{6}N(N+1)(2N+1) + \frac{1}{2}N(N+1) + O(N) \\
&= \frac{2}{3}N^3 + \frac{3}{2}N^2 + O(N) \\
&= \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x + O(x^{\frac{1}{2}})
\end{aligned}$$

2. Generalized mean-value about $a(n)$

Theorem 2 Let n be a positive integer, and $a(n) = [n^{\frac{1}{3}}]$,then

$$\sum_{n\leq x} a(n) = \sum_{n\leq x}[n^{\frac{1}{3}}] = \frac{3}{4}x^{\frac{4}{3}} + \frac{5}{2}x + \frac{11}{4}x^{\frac{2}{3}} + O(x^{\frac{1}{3}})$$

Proof $\sum\limits_{n\leq x} a(n) = \sum\limits_{n\leq x}[n^{\frac{1}{3}}]$

$$= \sum_{1^3\leq i<2^3} [i^{\frac{1}{3}}] + \sum_{2^3\leq i<3^3} [i^{\frac{1}{3}}] + \cdots + \sum_{N^3\leq i\leq x<(N+1)^3} [i^{\frac{1}{3}}] + O(N)$$

$$= 7\cdot 1 + 19\cdot 2 + \cdots + [(N+1)^3 - N^3]\cdot N + O(N)$$

$$= \sum_{j\leq N}[(j+1)^3 - j^3] + O(N)$$

$$= 3\sum_{j\leq N} j^3 + 3\sum_{j\leq N} j^2 + \sum_{j\leq N} j + O(N)$$

$$= 3[\frac{1}{2}N(N+1)]^2 + 3\cdot\frac{1}{6}N(N+1)(2N+1) + \frac{1}{2}N(N+1) + O(N)$$

$$= \frac{3}{4}N^4 + \frac{5}{2}N^3 + \frac{11}{4}N^2 + O(N)$$

$$= \frac{3}{4}x^{\frac{4}{3}} + \frac{5}{2}x + \frac{11}{4}x^{\frac{2}{3}} + O(x^{\frac{1}{3}})$$

Genarally,we have the following

Theorem 3 Let n be a positive integer,and $a(n) = [n^{\frac{1}{k}}]$,then

$$\sum_{n\leq x} a(n) = \sum_{n\leq x}[n^{\frac{1}{k}}] = \frac{k}{k+1}x^{\frac{k+1}{k}} + O(x)$$

Proof $\sum\limits_{n\leq x} a(n) = \sum\limits_{n\leq x}[n^{\frac{1}{k}}]$

$$= \sum_{1^k\leq i<2^k} [i^{\frac{1}{k}}] + \sum_{2^k\leq i<3^k} [i^{\frac{1}{k}}] + \cdots + \sum_{N^k\leq i\leq x<(N+1)^k} [i^{\frac{1}{k}}] + O(N)$$

$$= \sum_{j\leq N}[(j+1)^k - j^k] + O(N)$$

$$= \sum_{j\leq N}\begin{pmatrix} k \\ 1 \end{pmatrix}(j^k + \sum_{j\leq N}\begin{pmatrix} k \\ 2 \end{pmatrix} j^{j-k} + \cdots + \sum_{j\leq N}\begin{pmatrix} k \\ k \end{pmatrix} j + O(N)$$

$$= \sum_{j\le N}\sum_{l=1}^{k}\left(\begin{array}{c} k \\ l \end{array}\right) j^{k-l+1} + O(N)$$

$$= k \cdot \frac{N^{k+1}}{N+1} + O(N^k) + O(N)$$

$$= \frac{k}{k+1} x^{\frac{k+1}{k}} + O(x)$$

If we generaliz it from other view ,we can also have

Theorem 4 Let n be a positive integer,and $b(n) = (a(n))^2 = [\sqrt{n}\ ]^2$,then

$$\sum_{n\le x} b(n) = \sum_{n\le x} [\sqrt{n}\ ]^2 = \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{4}{3}} + O(x)$$

Proof $\sum_{n\le x} b(n) = \sum_{n\le x} [\sqrt{n}\ ]^2$

$$= \sum_{1^2\le i<2^2} [\sqrt{i}\ ]^2 + \sum_{2^2\le i<3^2} [\sqrt{i}\ ]^2 + \cdots + \sum_{N^2\le i\le x<(N+1)^2} [\sqrt{i}\ ]^2 + O(N^2)$$

$$= 3\cdot 1 + 5\cdot 4 + \cdots + [(N+1)^2 - N^2]N^2 + O(N^2)$$

$$= \sum_{j\le N} [(j+1)^2 - j^2]j^2 + O(N^2)$$

$$= 2\sum_{j\le N} j^3 + \sum_{j\le N} j^2 + O(N^2)$$

$$= 2\cdot [\frac{1}{2}N(N+1)]^2 + \frac{1}{6}[N(N+1)(2N+1)] + O(N^2)$$

$$= \frac{1}{2}N^4 + \frac{4}{3}N^3 + O(N^2)$$

$$= \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + O(x)$$

Theorem 5 Let n be a positive integer,and $b(n) = (a(n))^3 = [\sqrt{n}\ ]^3$,then

$$\sum_{n\le x} b(n) = \sum_{n\le x} [\sqrt{n}\ ]^3 = \frac{2}{5}x^{\frac{5}{2}} + \frac{5}{4}x^2 + O(x^{\frac{3}{2}})$$

Proof $\sum_{n\le x} b(n) = \sum_{n\le x} [\sqrt{n}\ ]^3$

$$= \sum_{1^2\le i<2^2} [\sqrt{i}\ ]^3 + \sum_{2^2\le i<3^2} [\sqrt{i}\ ]^3 + \cdots + \sum_{N^2\le i\le x<(N+1)^2} [\sqrt{i}\ ]^2 + O(N^3)$$

$$= 3\cdot 1 + 5\cdot 8 + \cdots + [(N+1)^2 - N^2]N^3 + O(N^3)$$

$$= \sum_{j\le N} [(j+1)^2 - j^2]j^3 + O(N^3)$$

$$= 2\sum_{j\le N} j^4 + \sum_{j\le N} j^3 + O(N^3)$$

$$= 2 \cdot \frac{1}{30} N(N+1)(2N+1)(3N^2+3N-1) + [\frac{1}{2}[N(N+1)]^2 + O(N^3)$$
$$= \frac{2}{5}N^5 + \frac{5}{4}N^4 + O(N^3)$$
$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{5}{4}x^2 + O(x^{\frac{3}{2}})$$

Theorem 6 Let n be a positive integer,and $b(n) = (a(n))^k = [\sqrt{n}\ ]^k$,then

$$\sum_{n \le x} b(n) = \sum_{n \le x} [\sqrt{n}\ ]^k = \frac{2}{k+2} x^{\frac{k+2}{2}} + O(x^{\frac{k+1}{2}})$$

Proof $\sum_{n \le x} b(n) = \sum_{n \le x} [\sqrt{n}\ ]^k$

$$= \sum_{1^2 \le i < 2^2} [\sqrt{i}\ ]^k + \sum_{2^2 \le i < 3^2} [\sqrt{i}\ ]^k + \cdots + \sum_{N^2 \le i \le x < (N+1)^2} [\sqrt{i}\ ]^k + O(N^k)$$
$$= 3 \cdot 1^k + 5 \cdot 2^k + \cdots + [(N+1)^2 - N^2]N^k + O(N^k)$$
$$= \sum_{j \le N} [(j+1)^2 - j^2] j^k + O(N^k)$$
$$= 2 \sum_{j \le N} j^{k+1} + \sum_{j \le N} j^k + O(N^k)$$
$$= 2 \cdot \frac{N^{k+2}}{k+2} + O(N^{k+1})$$
$$= \frac{2}{k+2} x^{\frac{k+2}{2}} + O(x^{\frac{k+1}{2}})$$

References

[1] F.Smarandache,Only problems not solutions.Chicago:Xiquan Publishing House,1993,74.