On the 80th Problem of F.Smarandache(II)

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Abstract The main purpose of this paper is to study the first power mean of $d(a(n))$; $\varphi(a(n))$ and their generations,and a sery of regular result is obtained,where $\varphi(n)$ is Euler totient funstion,$d(n)$ is divisor function and $a(n)$ is the sequence in problem 80 of "only problems not solutions" which was presented by professor F.Smarandache.

Keywords number-theoretic function; mean-value; asymptotic formula

In 1993,professor F.Smarandache presented 100 unsolved problems in [1],it arose great interests for scholars.Among them,the 80th problem is:

Square root: 0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6, 6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,8,8,$\cdots$

Study this sequences.

We donote the sequence in problem 80 as $a(n)$,it is not difficult to show that $a(n) = [\sqrt{n}\ ]$,where$[x]$ is the maximal integer that is no more than x.

1. Mean-value of $d(a(n))$and it's generalization

Theorem 1 Let n be a positive integer,and $a(n) = [\sqrt{n}\ ], d(n)$ be divisor function,then

$$\sum_{n\leq x} d(a(n)) = \sum_{n\leq x} d([\sqrt{n}\ ]) = \frac{1}{2}x\log x + \left(2c-\frac{1}{2}\right)x + O(x^{\frac{3}{4}})$$

Where c is Euler's constant.

Proof $\sum\limits_{n\leq x} d(a(n)) = \sum\limits_{n\leq x} d([\sqrt{n}\ ])$

$$= \sum_{1^2\leq i<2^2} d([\sqrt{i}\ ]) + \sum_{2^2\leq i<3^2} d([\sqrt{i}\ ]) + \cdots + \sum_{N^2\leq i\leq x<(N+1)^2} d([\sqrt{i}\ ]) + O(N^{\varepsilon})$$

$$= 3\cdot d(1) + 5\cdot d(2) + \cdots + [(N+1)^2 - N^2]d(N) + O(N^{\varepsilon})$$

$$= \sum_{j\leq N} (2j+1)d(j) + O(N^{\varepsilon})$$

Let $A(N) = \sum\limits_{j\leq N} d(j) = N\log N + (2c-1)N + O(N^{\frac{1}{2}})^{[2]}, f(j) = 2j+1$,by Abel's identity[2],we have

$$\sum_{j\leq N} (2j+1)d(j) = A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)\mathrm{d}t$$

$$= [N\log N + (2c-1)N + O(N^{\frac{1}{2}})](2N+1) - A(1)f(1) - \int_1^N [t\log t - (2c-1)t + O(N^{\frac{1}{2}})]\cdot 2\mathrm{d}t$$

$$= 2N^2\log N + 2(2c-1)N^2 + O(N^{\frac{3}{2}}) - 2\int_1^N t\log t\mathrm{d}t - 2\int_1^N (2c-1)t\mathrm{d}t - 2\int_1^N O(t^{\frac{1}{2}})\mathrm{d}t$$

$$= 2N^2\log N - 2(2c-1)N^2 + O(N^{\frac{3}{2}}) - N^2\log N^2 + \frac{1}{2}N^2 - 2(2c-1)N^2 + O(N^{\frac{3}{2}})$$

$$= N^2\log N + \left(2c - \frac{1}{2}\right)N^2 + O(N^{\frac{3}{2}})$$

So

$$\sum_{j\le N} d(a(n)) = \sum_{j\le N}(2j+1)d(j) + O(N^{\varepsilon})$$

$$= N^2\log N + \left(2c - \frac{1}{2}\right)N^2 + O(N^{\frac{3}{2}}) + O(N^{\varepsilon})$$

$$= \frac{1}{2}x\log x + \left(2c - \frac{1}{2}\right)x + O(x^{\frac{3}{4}})$$

Similarly,we have

Theorem 2 Let n be a positive integer,and $a(n) = [n^{\frac{1}{3}}]$, $d(n)$ be divisor function,then

$$\sum_{n\le x} d(a(n)) = \sum_{n\le x} d([n^{\frac{1}{3}}]) = \frac{1}{3}x\log x + \left(2c - \frac{1}{3}\right)x + O(x^{\frac{5}{6}})$$

Where c is Euler's constant.

Proof $\sum_{n\le x} d(a(n)) = \sum_{n\le x} d([n^{\frac{1}{3}}])$

$$= \sum_{1^3\le i<2^3} d([i^{\frac{1}{3}}]) + \sum_{2^3\le i<3^3} d([i^{\frac{1}{3}}]) + \cdots + \sum_{N^3\le i\le x<(N+1)^3} d([i^{\frac{1}{3}}]) + O(N^{\varepsilon})$$

$$= 7\cdot d(1) + 19\cdot d(2) + \cdots + [(N+1)^3 - N^3]d(N) + O(N^{\varepsilon})$$

$$= \sum_{j\le N}(3j^2+3j+1)d(j) + O(N^{\varepsilon})$$

Let $A(N) = \sum_{j\le N} d(j) = N\log N + (2c-1)N + O(N^{\frac{1}{2}})$[2], $f(j) = 3j^2+3j+1$, similarly , we have

$$\sum_{j\le N}(3j^2+3j+1)d(j) = A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)\mathrm{d}t$$

$$= [N\log N + (2c-1)N + O(N^{\frac{1}{2}})](3N^2+3N+1) - \int_1^N [t\log t - (2c-1)t + O(t^{\frac{1}{2}})](6t+3)\mathrm{d}t$$

$$= 3N^3\log N + 3(2c-1)N^3 + O(N^{\frac{5}{2}}) + 3N^2\log N + 3(2c-1)N^2 + N\log N + (2c-1)N$$

$$-7(2c-1)N - \int_1^N 6t^2\log t\mathrm{d}t - \int_1^N 6(2c-1)t^2\mathrm{d}t + O\left(\int_1^N 6t^{\frac{3}{2}}\mathrm{d}t\right) - \int_1^N 3t\log t\mathrm{d}t - \int_1^N 3(2c-1)t\mathrm{d}t$$

Because

$$\int_1^N 6t^2\log t\mathrm{d}t = 2N^3\log N - \frac{2}{3}N^3 + c_1,$$

$$\int_1^N 6(2c-1)t^2 \mathrm{d}t = 2(2c-1)N^3 + c_2,$$

$$\int_1^N 3t\log t \mathrm{d}t = \frac{3}{2}N^2\log N - \frac{3}{4}N^2 + c_3,$$

So

$$\sum_{j\le N}(3j^2+3j+1)d(j) = 3N^3\log N + 3(2c-1)N^3 - 2N^3\log N + \frac{2}{3}N^3 - 2(2c-1)N^3 + O(N^{\frac{5}{2}})$$

$$= N^3\log N + \left(2c - \frac{1}{3}\right)N^3 + O(N^{\frac{5}{2}})$$

As a result,we have

$$\sum_{j\le N} d(a(n)) = \sum_{j\le N} d([n^{\frac{1}{3}}])$$
$$= \sum_{j\le N}(3j^2+3j+1)d(j) + O(N^{\varepsilon})$$
$$= N^3\log N + \left(2c-\frac{1}{3}\right)N^3 + O(N^{\frac{5}{2}}) + O(N^{\varepsilon})$$
$$= \frac{1}{3}x\log x + \left(2c-\frac{1}{3}\right)x + O(x^{\frac{5}{6}})$$

Theorem 3 Let n be a positive integer , and $a(n) = [n^{\frac{1}{k}}], d(n)$ be divisor function,then

$$\sum_{n\le x} d(a(n)) = \sum_{n\le x} d([n^{\frac{1}{k}}]) = \frac{1}{k}x\log x + O(x)$$

Proof $\sum_{n\le x} d(a(n)) = \sum_{n\le x} d([n^{\frac{1}{k}}])$

$$= \sum_{1^k\le i<2^k} d([i^{\frac{1}{k}}]) + \sum_{2^k\le i<3^k} d([i^{\frac{1}{k}}]) + \cdots + \sum_{N^k\le i\le x<(N+1)^k} d([i^{\frac{1}{k}}]) + O(N^{\varepsilon})$$
$$= (2^k-1)d(1) + (3^k-2^k)d(2) + \cdots + [(N+1)^k - N^k]d(N) + O(N^{\varepsilon})$$
$$= \sum_{j\le N}[(j+1)^k - j^k]d(j) + O(N^{\varepsilon})$$

Let $A(N) = \sum_{j\le N} d(j) = N\log N + (2c-1)N + O(N^{\frac{1}{2}})$[2]$, f(j) = [(j+1)^k - j^k]$,then

$$\sum_{j\le N}[(j+1)^k - j^k]d(j) = A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)\mathrm{d}t$$
$$= [N\log N + (2c-1)N + O(N^{\frac{1}{2}})][(N+1)^k - N^k] - A(1)f(1)$$

$$-\int_1^N [t\log t+(2c-1)t+O(t^{\frac{1}{2}})](k(t+1)^{k-1}-kt^{k-1})\mathrm{d}t$$

$$=[N\log N+(2c-1)N+O(N^{\frac{1}{2}})](\sum_{l=1}^{k}\begin{pmatrix}k\\ l\end{pmatrix}N^{k-l})$$

$$-k\int_1^N [t\log t-2(2c-1)t+O(t^{\frac{1}{2}})](\sum_{l=1}^{k-1}\begin{pmatrix}k-1\\ l\end{pmatrix}t^{k-l-1})\mathrm{d}t$$

$$=\begin{pmatrix}k\\ 1\end{pmatrix}N^k\log N-\begin{pmatrix}k-1\\ 1\end{pmatrix}\int_1^N kt^{k-1}\log k\mathrm{d}t+O(N^k)$$

$$=\begin{pmatrix}k\\ 1\end{pmatrix}N^k\log N-\begin{pmatrix}k-1\\ 1\end{pmatrix}N^k\log N+O(N^k)$$

$$=N^k\log N+O(N^k)$$

So

$$\begin{aligned}\sum_{n\le x} d(a(n)) &= \sum_{n\le x} d([n^{\frac{1}{k}}])\\ &=\sum_{j\le N}[(j+1)^k-j^k]d(j)+O(N^{\varepsilon})\\ &=N^k\log N+O(N^k)+O(N^{\varepsilon})\\ &=\frac{1}{k}x\log x+O(x)\end{aligned}$$

2. Mean-value of $\varphi(a(n))$ and it's generalization

Theorem 4 Let n be a positive integer,and $a(n)=[\sqrt{n}\,]$, $\varphi(n)$ be Euler totient function,then

$$\sum_{n\le x}\varphi(a(n))=\sum_{n\le x}\varphi([\sqrt{n}\,])=\frac{4}{\pi^2}x^{\frac{3}{2}}+O(x\log x)$$

Proof $\sum_{n\le x}\varphi(a(n))=\sum_{n\le x}\varphi([\sqrt{n}\,])$

$$=\sum_{1^2\le i<2^2}\varphi([\sqrt{i}\,])+\sum_{2^2\le i<3^2}\varphi([\sqrt{i}\,])+\cdots+\sum_{N^2\le i\le x<(N+1)^2}\varphi([\sqrt{i}\,])+O(N)$$

$$=3\varphi(1)+5\varphi(2)+\cdots+[(N+1)^2-N^2]\varphi(N)+O(N)$$

$$=\sum_{j\le N}(2j+1)\varphi(j)+O(N)$$

Let $A(N)=\sum_{j\leq N}\varphi(j)=\frac{3}{\pi^2}N^2+O(N\log N)$[2], $f(j)=2j+1$,then

$$\sum_{j\leq N}(2j+1)\varphi(j)=A(N)f(N)-A(1)f(1)-\int_1^N A(t)f'(t)\mathrm{d}t$$

$$=\left[\frac{3}{\pi^2}N^2+O(N\log N)\right](2N+1)-\int_1^N\left[\frac{3}{\pi^2}t^2+O(t\log t)\right]2\mathrm{d}t$$
$$=\frac{6}{\pi^2}N^3+O(N^2\log N)-\frac{2}{\pi^2}N^3+O(N^2\log N)$$
$$=\frac{4}{\pi^2}N^3+O(N^2\log N)$$

Then

$$\begin{aligned}\sum_{n\leq x}\varphi([\sqrt{n}\,]) &=\sum_{j\leq N}(2j+1)\varphi(j)+O(N)\\ &=\frac{4}{\pi^2}N^3+O(N^2\log N)+O(N)\\ &=\frac{4}{\pi^2}x^{\frac{3}{2}}+O(x\log x)\end{aligned}$$

Similarly,we have

Theorem 5 Let n be a positive integer,and $a(n)=[n^{\frac{1}{3}}]$, $\varphi(n)$ be Euler totient function,then

$$\sum_{n\leq x}\varphi(a(n))=\sum_{n\leq x}\varphi([n^{\frac{1}{3}}])=\frac{9}{2\pi^2}x^{\frac{4}{3}}+O(x\log x)$$

Proof $\sum_{n\leq x}\varphi(a(n))=\sum_{n\leq x}\varphi([n^{\frac{1}{3}}])$

$$=\sum_{1^3\leq i<2^3}\varphi([i^{\frac{1}{3}}])+\sum_{2^3\leq i<3^3}\varphi([i^{\frac{1}{3}}])+\cdots+\sum_{N^3\leq i\leq x<(N+1)^3}\varphi([i^{\frac{1}{3}}])+O(N)$$
$$=7\varphi(1)+9\varphi(2)+\cdots+[(N+1)^3-N^3]\varphi(N)+O(N)$$
$$=\sum_{j\leq N}(3j^2+3j+1)\varphi(j)+O(N)$$

Let $A(N)=\sum_{j\leq N}\varphi(N)=\frac{3}{\pi^2}N^2+O(N\log N)$[2], $f(j)=3j^2+3j+1$,then

$$\sum_{j\leq N}(3j^2+3j+1)\varphi(j)=A(N)f(N)-A(1)f(1)-\int_1^N A(t)f'(t)\mathrm{d}t$$
$$=\left[\frac{3}{\pi^2}N^2+O(N\log N)\right](3N^2+3N+1)-\int_1^N\left[\frac{3}{\pi^2}t^2+O(t\log t)\right](6t+3)\mathrm{d}t$$
$$=\frac{9}{\pi^2}N^4-\frac{9}{2\pi^2}N^4+O(N^3\log N)$$

So

$$\sum_{n\le x}\varphi([i^{\frac{1}{3}}]) = \sum_{j\le N}(3j^2+3j+1)\varphi(j)+O(N)$$
$$= \frac{9}{2\pi^2}N^4 + O(N^3\log N) + O(N)$$
$$= \frac{9}{2\pi^2}x^{\frac{4}{3}} + O(x\log x)$$

Theorem 6 Let n be a positive integer,and $a(n) = [n^{\frac{1}{k}}], \varphi(n)$ be Euler totient function,then

$$\sum_{n\le x}\varphi(a(n)) = \sum_{n\le x}\varphi([n^{\frac{1}{k}}]) = \frac{6k}{(k+1)\pi^2}x^{\frac{k+1}{k}} + O(x\log x)$$

Proof $\sum_{n\le x}\varphi(a(n)) = \sum_{n\le x}\varphi([n^{\frac{1}{k}}])$

$$= \sum_{1^k\le i<2^k}\varphi([i^{\frac{1}{k}}]) + \sum_{2^k\le i<3^k}\varphi([i^{\frac{1}{k}}]) + \cdots + \sum_{N^k\le i\le x<(N+1)^k}\varphi([i^{\frac{1}{k}}]) + O(N)$$
$$= \sum_{j\le N}[(j+1)^k - j^k]\varphi(j) + O(N)$$

Let $A(N) = \sum_{j\le N}\varphi(j) = \frac{3}{\pi^2}N^2 + O(N\log N)^{[2]}, f(j) = [(j+1)^k - j^k]$,then

$$\sum_{j\le N}[(j+1)^k - j^k]\varphi(j) = A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)\mathrm{d}t$$
$$= \left[\frac{3}{\pi^2}N^2 + O(N\log N)\right][(N+1)^k - N^k] - \int_1^k\left[\frac{3}{\pi^2}t^2 + O(t\log t)\right]k[(t+1)^{k-1} - t^{k-1}]\mathrm{d}t$$
$$= \frac{3k}{\pi^2}N^{k+1} + O(N^k\log N) - \frac{k(k-1)}{k+1}\frac{3}{\pi^2}N^{k+1}$$
$$= \frac{6k}{(k+1)\pi^2}N^{k+1} + O(N^k\log N)$$

So

$$\sum_{n\le x}\varphi(a(n)) = \sum_{n\le x}\varphi([n^{\frac{1}{k}}])$$
$$= \sum_{j\le N}[(j+1)^k - j^k]\varphi(j) + O(N)$$
$$= \frac{6k}{(k+1)\pi^2}N^{k+1} + O(N^k\log N) + O(N)$$
$$= \frac{6k}{(k+1)\pi^2}x^{\frac{k+1}{k}} + O(x\log x)$$

References

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[2] T.M.Apostol,Introduction to analytic number theory.New York:Springer-Verlag,1976.