ON THE CALCULUS OF SMARANDACHE FUNCTION

by

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Introduction. The Smarandache function $S : \mathbf{N}^* \longrightarrow \mathbf{N}^*$ is defined [5] by the condition that $S(n)$ is the smallest integer m such that $m!$ is divisible by n. So, we have $S(1) = 1, S(2^{12}) = 16$.

Considering on the set $\mathbf{N}^*$ two laticeal structures $\mathcal{N} = (\mathbf{N}^*, \wedge, \vee)$ and $\mathcal{N}_d = (\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee})$, where $\wedge = \min, \vee = \max$, $\underset{d}{\wedge}=$ the greatest common divisor, $\overset{d}{\vee}=$ the smallest common multiple, it results that S has the followings properties:

$$(s_1) \quad S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$
$$(s_2) \quad n_1 \leq_d n_2 \Longrightarrow S(n_1) \leq S(n_2)$$

where $\leq$ is the order in the lattice $\mathcal{N}$ and $\leq_d$ is the order in the lattice $\mathcal{N}_d$. It is said that

$$n_1 \leq_d n_2 \Longleftrightarrow n_1 \text{ divides } n_2$$

From these properties we deduce that in fact on must consider

$$S : \mathcal{N}_d \longrightarrow \mathcal{N}$$

Methods for the calculus of S. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, from (s_1) it results

$$S(n) = \vee S(p_i^{\alpha_i})$$

so the calculus of $S(n)$ is reduced to the calculus of $S(p^\alpha)$.

If $e_p(n)$ is the exponent of the prime p in the decomposition into primes of $n!$:

$$n! = \prod_{j=1}^{t} p_j^{e_{p_j}(n)}$$

by Legendre's formula it is said that

$$e_p(n) = \sum_{i \geq 1} [\frac{n}{p^i}]$$

Also we have

$$e_p(n) = \frac{n - \sigma_{(p)}(n)}{p-1} \tag{2}$$

where $[x]$ is the integer part of x and $\sigma_{(p)}(n)$ is the sum of digits of n in the numerical scale

$$(p) : 1 \ , \ p \ , \ p^2 \ , ..., \ p^i ...$$

For the calculus of $S(p^a)$ we need to consider in addition a generalised numerical scale $[p]$ given by:

$$[p] : \ a_1(p) \ , \ a_2(p) \ , \dots , a_i(p) \ , \ \dots$$

where $a_i(p) = (p^i - 1)/(p-1)$.Then in [3] it is showed that

$$S(p^a) = p(\alpha_{[p]})_{(p)} \tag{3}$$

that is the value of $S(p^\alpha)$ is obtained multiplying p by the number obtained writing the exponent α in the generalised scale $[p]$ and "reading" it in the usual scale (p).

Let us observe that the calculus in the generalised scale $[p]$ is essentially different from the calculus in the scale (p).That is because if we note

$$b_n(p) = p^n$$

then for the usual scale (p) it results the recurence relation

$$b_{n+1}(p) = p \cdot b_n(p)$$

and for the generalised scale $[p]$ we have

$$a_{n+1}(p) = p \cdot a_n(p) + 1$$

For this,to add some numbers in the scale $[p]$ we do as follows:

1) We start to add from the digits of "decimals", that is from the column corresponding to $a_2(p)$.

2) If adding some digits it is obtained $pa_2(p)$, then we utilise an *unit* from the classe of "units" (the column corresponding to $a_1(p)$) to obtain $p \cdot a_2(p) + 1 = a_3(p)$. Continuing to add, if agains it is obtained $p \cdot a_2(p)$, then a new unit must be used from the classe of units, etc.

Example. If

$$m_{[5]} = 442 = 4a_3(5) + 4a_2(5) + 2a_1(5)\ ,\ n_{[5]} = 412\ ,\ r_{[5]} = 44$$

then

$$\begin{array}{rcr} m+n+r & = & 442\ + \\ & & 412 \\ & & 44 \\ & & dcba \end{array}$$

To find the digits a, b, c, d we start to add from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, if we take an unit from the first column we get:

$$5a_2(5) + 4a_2(5) + 1 = a_3(5) + 4a_2(5)$$

so $b = 4$.

Continuing the addition we have:

$$4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$$

and using a new unit (from the first column) it results:

$$4a_3(5) + 4a_3(5) + a_3(5) + 1 = a_4(5) + 4a_3(5)$$

so $c = 4$ and $d = 1$.

Finaly, adding the remained units:

$$4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$$

it results that the digit $b = 4$ must be changed in $b = 5$ and $a = 0$.

So

$$m_{[5]} + n_{[5]} + r_{[5]} = 1450_{[5]} = a_4(5) + 4a_3(5) + 5a_2(5)$$

Remarque. As it is showed in [5],writing a positive integer α in the scale $[p]$ we may find the first non-zero digit on the right equals to p.Of course,that is no possible in the standard scale (p).

Let us return now to the presentation of the formulae for the calculus of the Smarandache function. For this we expresse the exponent α in both the scales (p) and $[p]$:

$$\alpha_{(p)} = c_u p^u + c_{u-1}p^{u-1} + \ldots + c_1 p + c_0 = \sum_{i=0}^{u} c_i p^i \tag{4}$$

and

$$\alpha_{[p]} = k_v a_v(p) + k_{v-1}a_{v-1}(p) + \ldots + k_1 a_1(p) = \sum_{j=1}^{v} k_j a_j(p) =$$
$$= \sum_{j=1}^{v} k_j \frac{p^j - 1}{p-1}$$

It results

$$(p-1)\alpha = \sum_{j=1}^{v} k_j p^j - \sum_{j=1}^{v} k_j \tag{5}$$

so,because $\sum_{j=1}^{v} k_j p^j = p(\alpha_{[p]})_{(p)}$, we get:

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{6}$$

From (4) we deduce

$$p\alpha = \sum_{i=0}^{u} c_i(p^{i+1} - 1) + \sum_{i=0}^{u} c_i$$

and

$$\frac{p}{p-1}\alpha = \sum_{i=0}^{u} c_i a_{i+1}(p) + \frac{1}{p-1}\sigma_{(p)}(\alpha)$$

Consequently

$$\alpha = \frac{p-1}{p}(\alpha_{(p)})_{[p]} + \frac{1}{p}\sigma_{(p)}(\alpha) \tag{7}$$

Replacing this expression of α in (6) we get:

$$S(p^\alpha) = \frac{(p-1)^2}{p}(\alpha_{(p)})_{[p]} + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \tag{8}$$

Example. To find $S(3^{89})$ we shall utilise the equality (3). For this we have:

$$\begin{array}{l} (3) \;:\; 1 \;,\; 3 \;,\; 9 \;,\; 27 \;,\; 81 \;,\; \ldots \\ [3] \;:\; 1 \;,\; 4 \;,\; 13 \;,\; 40 \;,\; 121 \;,\; \ldots \end{array}$$

and $89_{[3]} = 2021$, so $S(3^{89}) = 3(2021)_{(3)} = 183$. That is 183! is divisible by 3^{89} and it is the smallest factorial with this property.

We shall use now the equality (6) to calculate the same value $S(3^{89})$. For this we observe that $\sigma_{[3]}(89) = 5$ and, so $S(3^{89}) = 2 \cdot 89 + 5 = 183$.

Using (8) we get $89_{(3)} = 10022$ and :

$$S(3^{89}) = \frac{4}{3}(10022)_{[3]} + \frac{2}{3} \cdot 5 + 5 = 183$$

It is possible to expresse $S(p^\alpha)$ by mins of the exponent $e_p(\alpha)$ in the following way: from (2) and (7) it results

$$e_p(\alpha) = (\alpha_{(p)})_{[p]} - \alpha \tag{9}$$

and then from (8) and (9) it results

$$S(p^\alpha) = \frac{(p-1)^2}{p}(e_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \tag{10}$$

Remarque. From (3) and (8) we deduce a connection between the integer α writen in the scale $[p]$ and readed in the scale (p) and the same integer writed in the scale (p) and readed in the scale $[p]$. Namely:

$$p^2(\alpha_{[p]})_{(p)} - (p-1)^2(\alpha_{(p)})_{[p]} = p\sigma_{[p]}(\alpha) + (p-1)\sigma_{(p)}(\alpha) \tag{11}$$

The function $i_p(\alpha)$. In the followings let we note $S(p^\alpha) = S_p(\alpha)$. Then from Legendre's formula it results:

$$(p-1)\alpha < S_p(\alpha) \leq p\alpha$$

that is $S(p^\alpha) = (p-1)\alpha + x = p\alpha - y$.

From (6) it results that $x = \sigma_{[p]}(\alpha)$ and to find y let us write $S_p(\alpha)$ under the forme

$$S_p(\alpha) = p(\alpha - i_p(\alpha)) \tag{12}$$

As it is showed in [4] we have $0 \leq i_p(\alpha) \leq [\frac{\alpha-1}{p}]$.

Then it results that for each function S_p there exists a function i_p so that we have the linear combination

$$\frac{1}{p}S_p(\alpha) + i_p(\alpha) = \alpha \tag{13}$$

In [1] it is proved that

$$i_p(\alpha) = \frac{\alpha - \sigma_{[p]}(\alpha)}{p} \tag{14}$$

and so it is an evident analogy between the expression of $e_p(\alpha)$ given by the equality (2) and the expression of $i_p(\alpha)$ in (14).

In [1] it is also showed that

$$\alpha = (\alpha_{[p]})_{(p)} + [\frac{\alpha}{p}] - [\frac{\sigma_{[p]}(\alpha)}{p}] = (\alpha_{[p]})_{(p)} + \frac{\alpha - \sigma_{[p]}(\alpha)}{p}$$

and so

$$S(p^\alpha) = p(\alpha - [\frac{\alpha}{p}] + [\frac{\sigma_{[p]}(\alpha)}{p}]) \tag{15}$$

Finaly,let us observe that from the definition of Smarandache function it results that

$$(S_p \circ e_p)(\alpha) = p[\frac{\alpha}{p}] = \alpha - \alpha_p$$

where α_p is the remainder of α modulus p.Also we have

$$(e_p \circ S_p)(\alpha) \geq \alpha \text{ and } e_p(S_p(\alpha) - 1) < \alpha$$

so using (2) it results

$$\frac{S_p(\alpha) - \sigma_{(p)}(S_p(\alpha))}{p-1} \geq \alpha \text{ and } \frac{S_p(\alpha) - 1 - \sigma_{(p)}(S_p(\alpha) - 1)}{p-1} < \alpha$$

Using (6) we obtains that $S(p^\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x) \leq \sigma_{[p]}(\alpha) \leq \sigma_{(p)}(x-1)+1$$

The calculus of card(S^{-1}(n)).Let $q_1, q_2, ..., q_k$ be all the prime itegers smallest then n and non dividing n. Let also denote shortly $e_{q_j}(n) = f_j$.A solution x_0 of the equation

$$S(x) = n$$

has the property that x_0 divides $n!$ and non divides $(n-1)!$.Now, if $d(n)$ is the number of positive divisors of n,from the inclusion

$$\{m \ / \ m \text{ divides } (n-1)!\} \subset \{m \ / \ m \text{ divides } n!\}$$

and using the definition of Smarandache function it results that

$$\text{card}(S^{-1}(n)) = d(n!) - d((n-1)!) \tag{16}$$

Example. In [6] A. Stuparu and D. W. Sharpe has proved that if p is a given prime,the equation

$$S(x) = p$$

has just $d((p-1)!)$ solutions (all of them in between p and $p!$) .Let us observe that $e_p(p!) = 1$ and $e_p((p-1)!) = 0$,so because

$$d(p!) = (e_p(p!)+1)(f_1+1)(f_2+1)...(f_k+1) = 2(f_1+1)(f_2+1)...(f_k+1)$$
$$d((p-1)!) = (f_1+1)(f_2+1)...(f_k+1)$$

it results

$$\text{card}(S^{-1}(p!)) = d(p!) - d((p-1)! = d((p-1)!)$$

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