ON THE CONVERGENCE OF THE ERDOS HARMONIC SERIES

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The purpose of this article is to study the convergence of a few series with the Erdos function. The work is based on results concerning the convergence of some series with the Smarandache function.

1. INTRODUCTION

The results used in this article are presented briefly in the following. These concern the relationship between the Smarandache and the Erdos functions and the convergence of some series. These two functions are important function in Number Theory. They are defined as follows:

- The Smarandache function [Smarandache, 1980] is $S: N^* \to N$ defined by

$$S(n) = \min\{k \in N | k! = Mn\} \ (\forall n \in N^*). \tag{1}$$

- The Erdos function is $P: N^* \to N$ defined by

$$P(n) = \min\{p \in N \mid n = Mp \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}), \ P(1) = 0. \tag{2}$$

The main properties of them are:

$$(\forall a,b \in N^*) \ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}, \ P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*) \ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim.} \tag{4}$$

Erdos [1991] found the relationship between these two functions that is given by

$$\lim_{n\to\infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0. \tag{5}$$

This important result was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}}, \text{ where } \lim_{n\to\infty} a_n = 0. \tag{6}$$

Equations (5-6) are very important because allow us to translate convergence properties on the Smarandache function to convergence properties on the Erdos function. This translation represents the main technique that is used to obtain the convergence of some series with the function P.

2. THE ERDOS HARMONIC SERIES

The Erdos harmonic series can be defined by $\sum_{n\geq 2}\frac{1}{P^a(n)}$. This is one of the important series with the Erdos function and its convergence is studied starting from the convergence of the Smarandache harmonic series $\sum_{n\geq 2}\frac{1}{S^a(n)}$. Some results concerning series with the function S are reviewed briefly in the following:

- If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = \infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent. [Cojocaru, 1997].

  (7)
- The series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is divergent. [Tabirca, 1998]

  (8)
- The series $\sum_{n\geq 2}\frac{1}{S^a(n)}$ is divergent for all a>0. [Luca, 1999] (9)

These above results are translated to the similar properties on the Erdos function.

Theorem 1. If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = \infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{P(x_n)}$ is divergent.

Proof The proof is obvious based on the equation $P(x_n)\leq S(x_n)$. Therefore, the equation $\frac{x_{n+1}-x_n}{P(x_n)}\geq\frac{x_{n+1}-x_n}{S(x_n)}$ and the divergence of the series $\sum_{n>1}\frac{x_{n+1}-x_n}{S(x_n)}$ give that the series $\sum_{n>1}\frac{x_{n+1}-x_n}{P(x_n)}$ is divergent.

♦

A direct consequence of Theorem 1 is the divergence of the series $\sum_{n>1} \frac{1}{P(a \cdot n + b)}$, where $a, b>0$ are positive numbers. This gives that $\sum_{n\geq 2} \frac{1}{P(n)}$ is divergent and moreover that $\sum_{n\geq 2} \frac{1}{P^a(n)}$ is divergent for all $a<1$.

Theorem 2. The series $\sum_{n\geq 2} \frac{1}{P^a(n)}$ is divergent for all $a>1$.

Proof The proof studies two cases.

Case 1. $a \geq \frac{1}{2}$.

In this case, the proof is made by using the divergence of $\sum_{n\geq 2} \frac{1}{S^a(n)}$.

Denote $A = \{i = \overline{2,n} \mid S(i) = P(i)\}$ and $B = \{i = \overline{2,n} \mid S(i) > P(i)\}$ a partition of the set $\{i = \overline{1,n}\}$. We start from the following simple transformation

$$\sum_{i=2}^{n} \frac{1}{P^a(i)} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \left[\frac{1}{P^a(i)} - \frac{1}{S^a(i)}\right] = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \frac{S^a(i) - P^a(i)}{P^a(i)\cdot S^a(i)}. \tag{10}$$

An $i \in B$ satisfies $S^a(i) - P^a(i) \geq 1$ and $P(i) < S(i) \leq n$ thus, (10) becomes

$$\sum_{i=2}^{n} \frac{1}{P^a(i)} \geq \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \frac{1}{n^{2\cdot a}} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \frac{1}{n^{2\cdot a}} \cdot |B|. \tag{11}$$

The series $\sum_{n\geq 2} \frac{1}{P^a(n)}$ is divergent because the series $\sum_{n\geq 2} \frac{1}{S^a(n)}$ is divergent and

$$\lim_{n\to\infty} \frac{|B|}{n^{2\cdot a}} = \lim_{n\to\infty} \frac{n \cdot e^{-(\sqrt{2}+a_n)\cdot\sqrt{\ln n \cdot \ln \ln n}}}{n^{2\cdot a}} = \lim_{n\to\infty} \frac{1}{n^{2\cdot a-1} \cdot e^{(\sqrt{2}+a_n)\cdot\sqrt{\ln n \cdot \ln \ln n}}} = 0.$$

Case 2. $\frac{1}{2} > a > 1$.

The first case gives that the series $\sum_{n\geq 2} \frac{1}{P^{\frac{1}{2}}(n)}$ is divergent.

Based on $P^{\frac{1}{2}}(n) > P^{a}(n)$, the inequality $\sum_{i=2}^{n} \frac{1}{P^{a}(i)} > \sum_{i=2}^{n} \frac{1}{P^{\frac{1}{2}}(i)}$ is found. Thus, the series $\sum_{n \geq 2} \frac{1}{S^{a}(n)}$ is divergent. ♦

The technique that has been applied to the proof of Theorem 2 can be used in the both ways. Theorem 2 started from a property of the Smarandache function and found a property of the Erdos function. Opposite, Finch [1999] found the property $\lim_{n\to\infty} \frac{\sum_{i=2}^{n} \frac{\ln S(i)}{\ln i}}{n} = \lambda$ based on the similar property $\lim_{n\to\infty} \frac{\sum_{i=2}^{n} \frac{\ln P(i)}{\ln i}}{n} = \lambda$, where $\lambda=0.6243299$ is the Golomb-Dickman constant. Obviously, many other properties can be proved using this technique. Moreover, Equations (5-6) gives a very interesting fact - **"the Smarandache and Erdos function may have the same behavior on the convergence problems."**

References

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