# ON THE CONVERGENCE OF THE EULER HARMONIC SERIES

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***Abstract:*** *The aim of this article is to study the convergence of the Euler harmonic series. Firstly, the results concerning the convergence of the Smarandache and Erdos harmonic functions are reviewed. Secondly, the Euler harmonic series is proved to be convergent for a>1, and divergent otherwise. Finally, the sums of the Euler harmonic series are given.*

***Key words:*** ***series, convergence, Euler function.***

The purpose of this article is to introduce the Euler harmonic series and to study its convergence. This problem belongs to a new research direction in Number Theory that is represented by convergence properties of series made with the most used Number Theory functions.

## 1. Introduction

In this section, the important results concerning the harmonic series for the Smarandache and Erdos function are reviewed.

*Definition 1.* If $f : N \to N^*$ is a function, then the series $\sum_{n\geq 1} \frac{1}{f^a(n)}$ is the harmonic series associated to *f* and is shortly named the *f* harmonic series.

The convergence of this sort of series has been studied for the Smarandache and Erdos functions so far. Both are important functions in Number Theory being intensely studied. The definitions and main properties of these two important functions are presented in the following:

- The Smarandache function is $S: N^* \to N$ defined by

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$$S(n) = \min\{k \in N | k! = Mn\} \ (\forall n \in N^*). \tag{1}$$

- The Erdos function is $P : N^* \to N$ defined by

$$P(n) = \max\{p \in N \mid n = Mp \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}),\ P(1) = 0. \tag{2}$$

The main properties of them are:

$$(\forall a, b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\ P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*)\ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim}. \tag{4}$$

Erdos [1995] found the relationship between these two functions that is given by

$$\lim_{n \to \infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0. \tag{5}$$

The series $\sum_{n \ge 2} \frac{1}{S(n)}$ and $\sum_{n \ge 2} \frac{1}{P(n)}$ are obviously divergent from Equation (4).

The divergence of the series $\sum_{n \ge 2} \frac{1}{S^2(n)}$ was an open problem for more than ten years. Tabirca [1998] proved the its divergence using an analytical technique. Luca [1999] was able to prove the divergence of the series $\sum_{n \ge 2} \frac{1}{S^a(n)}$ refining Tabirca's technique. Thus, the Smarandache harmonic series $\sum_{n \ge 2} \frac{1}{S^a(n)}$, $a \in R$ is divergent. Based on this result and on Equation (5), Tabirca [1999] showed that the Erdos harmonic series $\sum_{n \ge 2} \frac{1}{P^a(n)}$, $a \in R$ is divergent too.

Unfortunately, this convergence property has not been studied for the Euler function. This function is defined as follow: $\varphi : N \to N$, $(\forall n \in N)\ \varphi(n) = \left|\left\{k = 1, 2, \dots, n \mid (k,n) = 1\right\}\right|$. The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

$$(\forall\, a, b \in N)(a,b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \text{ - the multiplicative property} \tag{6}$$

$$a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s} \Rightarrow \varphi(a) = a \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right) \tag{7}$$

$$(\forall\, a \in N) \sum_{d|a} \varphi(d) = a. \tag{8}$$

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

## 2. The Convergence of the Euler Harmonic Series

In this section, the problem of the convergence for the Euler harmonic series is totally solved. The Euler harmonic series $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$, $a \in R$ is proved to have the same behavior as the harmonic series $\sum_{n\geq 1}\frac{1}{n^a}$, $a \in R$.

*Proposition 1.* The series $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$ is divergent for $a \leq 1$.

**Proof**

The proof is based on the equation

$$\varphi(n) \leq n \left(\forall\ n \geq 1\right). \tag{9}$$

Since $\frac{1}{\varphi^a(n)} \geq \frac{1}{n^a} \left(\forall\ n \geq 1\right)$ and $\sum_{n\geq 1}\frac{1}{n^a}$ is divergent, it follows that $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$ is divergent too.

♦

The convergence of the series for $a>1$ is more difficult than the previous and is studied in the following.

Let us define the function $d : N^* \to N$ by $d(n) = \left|\{p \text{ prime} \mid n = Mp\}\right|$. The main properties of this function are given by the next proposition.

*Proposition 2.* The function $d$ satisfies the following equation:

a) $d(1)=0$. (10a.)

b) $\left(\forall\ a,b \in N^*\right)(a,b) = 1 \Rightarrow d(a\cdot b) = d(a) + d(b)$. (10b.)

c) $\left(\forall\ n \in N^*\right) d(n) \leq \log_2(n)$. (10c.)

**Proof**

Equation (10a.) is obvious.

Let $a = p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ and $b = q_1^{k_1}\cdot q_2^{k_2}\cdot\ldots\cdot q_t^{k_t}$ be the prime number decomposition of two relative prime numbers. Thus, $a\cdot b = p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}\cdot q_1^{k_1}\cdot q_2^{k_2}\cdot\ldots\cdot q_t^{k_t}$ gives the prime number decomposition for $ab$. Since the equation $d(a\cdot b) = s+t$, $d(a) = s$ and $d(b) = t$ hold in the above hypothesises, Equation (10b) is true.

Let $n = p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ be the prime number decomposition of $n$. Equation (10b) gives the following inequality

$$d(n) = d(p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}) = d(p_1^{m_1}) + d(p_2^{m_2}) + \ldots + d(p_s^{m_s}) = 1+1+\ldots+1 \leq$$
$$\leq \log_2\left(p_1^{m_1}\right) + \log_2\left(p_2^{m_2}\right) + \ldots + \log_2\left(p_s^{m_s}\right) = \log_2(p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}) = \log_2(n)$$

that proves Equation (10c). ♦

The following proposition proposes a new inequality concerning the Euler function.

*Proposition 3.* $(\forall\, n \geq 1)\ \varphi(n) \geq \dfrac{n}{1+\log_2 n}$.

**Proof**

Let $n = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}$ be the prime number decomposition of $n$ such that $p_1 < p_2 < \ldots < p_s$. Thus, $\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right)$ holds. Using the order $p_1 < p_2 < \ldots < p_s$, it follows that $2 \leq p_1, 3 \leq p_2, \ldots, d(n)+1 < p_s$. These inequalities are used as follows:

$$\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right) \geq$$

$$\geq n \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \ldots \cdot \left(1 - \frac{1}{d(n)+1}\right) = \frac{n}{d(n)+1}$$

Equation (10c) used in the last inequality gives $\varphi(n) \geq \dfrac{n}{1+\log_2 n}$. ♦

***Proposition 4.*** If $a>1$, then the series $\sum_{n\geq 1} \left(\frac{1+\log_2 n}{n}\right)^a$ is convergent.

**Proof**

The proof uses the following convergence test: *"if $(a_n)_{n>0}$ is a decreasing sequence, then the series $\sum_{n>0} a_n$ and $\sum_{n>0} 2^n \cdot a_{2^n}$ have the same convergence".*

Because the sequence $\left(\left(\frac{1+\log_2 n}{n}\right)^a\right)_{n>0}$ is decreasing, the above test can be applied. The condensed series is $\sum_{n\geq 1} 2^n \cdot \left(\frac{1+\log_2 2^n}{2^n}\right)^a = \sum_{n\geq 1} \frac{(1+n)^a}{2^{n\cdot(a-1)}} = 2^{a-1} \cdot \sum_{n\geq 2} \frac{n^a}{2^{n\cdot(a-1)}}$ that is obviously convergent. ♦

***Theorem 5.*** If $a>1$, then the series $\sum_{n\geq 1} \frac{1}{\varphi^a(n)}$ is convergent.

**Proof**

According to Proposition 4, the series $\sum_{n\geq 1} \left(\frac{1+\log_2 n}{n}\right)^a$ is convergent. Proposition 3 gives the inequality $\left(\frac{1+\log_2 n}{n}\right)^a \geq \frac{1}{\varphi^a(n)}$, thus the series $\sum_{n\geq 1} \frac{1}{\varphi^a(n)}$ is convergent too. ♦

The interesting fact is that the Euler harmonic series has the same behaviour as the classical harmonic series. Therefore, both are convergent for $a>1$ and both are divergent for $a\leq 1$. The right question is to find information about the sum of the series in the convergence case. Let us denote $E(a)=\sum_{n\geq 1}\frac{1}{\varphi^{a}(n)}$ the sum of the Euler harmonic series for $a>1$. These constants can be computed by using a simple computation. They are presented in Table 1 for $a=2,3,\ldots,7$.

| $a$ | $E(a)$ | $a$ | $E(a)$ |
|---|---|---|---|
| 2 | 3.39049431 | 5 | 2.09837919 |
| 3 | 2.47619474 | 6 | 2.04796102 |
| 4 | 2.20815078 | 7 | 2.02369872 |

**Table 1.** The values for $E(a)$.

Unfortunately, none of the above constants are known. Moreover, a relationship between the classical constants ($\pi$, $e$, ...) and them are not obvious. Finding properties concerning the constants $E(a)$ still remains an open research problem.

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