ON THE DIVERGENCE OF THE SMARANDACHE HARMONIC SERIES

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For any positive integer n let $S(n)$ be the minimal positive integer m such that $n \mid m!$. In [3], the authors showed that

$$\sum_{n\geq 1} \frac{1}{S(n)^2} \tag{1}$$

is divergent and attempted, with limited succes, to gain information about the behaviour of the partial sum

$$A(x) = \sum_{n\leq x} \frac{1}{S(n)^2}$$

by comparing it with both $\log x$ and $\log x + \log\log x$.

In this note we show that none of these two functions is a suitable candidate for the order of magnitude of $A(x)$.

Here is the result. For any $\delta > 0$ and $x \geq 1$ denote by

$$A_\delta(x) = \sum_{n\leq x} \frac{1}{S(n)^\delta}, \tag{2}$$

$$B_\delta(x) = \log A_\delta(x). \tag{3}$$

Then,

Theorem.

For any $\delta > 0$,

$$B_\delta(x) \geq \log 2 \cdot \frac{\log x}{\log\log x} - o\Big(\frac{\log x}{\log\log x}\Big). \tag{4}$$

What the above theorem basically says is that for fixed δ and for arbitrary $\epsilon > 0$, there exists some constant C (depending on both δ and ϵ), such that

$$A_\delta(x) > 2^{(1-\epsilon)\frac{\log x}{\log\log x}} \qquad \text{for } x > C. \tag{5}$$

Notice that equation (5) asserts that $A_\delta(x)$ grows much faster than any polynomial in $\log(x)$, so one certainly shouldn't try to approximate it by a linear in $\log x$.

The Proof.

In [1], we showed that

$$\sum_{n\geq 1} \frac{1}{S(n)^\delta} \tag{6}$$

diverges for all $\delta > 0$. Since the argument employed in the proof is relevant for our purposes, we reproduce it here.

Let $t \geq 1$ be an integer and $p_1 < p_2 < \ldots < p_t$ be the first t prime numbers. Notice that any integer $n = p_t m$ where m is squarefree and all the prime factors of

m belong to $\{p_1, p_2, \ldots, p_{t-1}\}$ will certainly satisfy $S(n) = p_t$. Since there are at least 2^{t-1} such m's (the power of the set $\{p_1, \ldots, p_{t-1}\}$), it follows that series (6) is bounded below by

$$\sum_{t\geq 1} \frac{2^{t-1}}{p_t^\delta} = \sum_{t\geq 1} 2^{t-1-\delta \log_2 p_t}. \tag{7}$$

The argument ends noticing that since

$$\lim_{t\to\infty} \frac{p_t}{t \log t} = 1,$$

it follows that the exponent $t - 1 - \delta \log_2 p_t$ is always positive for t large enough. This proves the divergence of the series (6).

For the present theorem, the only new thing is the fact that we do not work with the whole series (6) but only with its partial sum $A_\delta(x)$. In particular, the parameter t from the above argument is precisely the maximal value of s for which $p_1 p_2 \ldots p_s \leq x$. In order to prove our theorem, we need to come up with a good lower bound on t.

We show that for all $\epsilon > 0$ one has

$$t > (1-\epsilon)\frac{\log x}{\log\log x} \tag{8}$$

provided that x is enough large. Assume that this is not so. It then follows that there exists some $\epsilon > 0$ such that

$$t < (1-\epsilon)\frac{\log x}{\log\log x} \tag{9}$$

for arbitrarily large values of x. Since t was the value of the maximal s such that $p_1 p_2 \ldots p_s \leq x$, it follows that

$$p_1 p_2 \ldots p_{t+1} > x. \tag{10}$$

From a formula in [2], it follows easily that

$$p_i \leq 2i \log i \qquad \text{for } i \geq 3. \tag{11}$$

It now follows, by taking logarithms in (10) and using (11), that

$$\log x < \sum_{i=1}^{t-1} \log p_i < C_1 + (t-1)\log 2 + \sum_{i=3}^{t+1} (\log i + \log\log i) <$$

$$C_1 + (t-1)\log 2 + \int_3^{t+2} (\log y + \log\log y)dy <$$

$$C_1 + (t-1)\log 2 + (t+2)(\log(t+2) + \log\log(t+2)), \tag{12}$$

where $C_1 = \log 6$. Since t can be arbitrarily large (because x is arbitrarily large), it follows that one can just work with

$$\log x < t(\log t + 2\log\log t). \tag{13}$$

Indeed, the amount $(t+2)(\log(t+2) + \log\log(t+2))$ appearing in the right hand side of (12) can be replaced by $t(\log t + \log\log t) + f(t)$ where $f(t) = 2\log t + 2\log\log t +$

$O(1)$ and then the sum of $f(t)$ with the linear term from the right hand side of (12) can certainly be bounded above by $t \log\log t$ for t large enough. Hence,

$$\log x < t(\log t + 2\log\log t). \tag{14}$$

Using inequality (9) to bound the factor t appearing in (14) in terms of x and the obvious inequality

$$t \leq (1-\epsilon)\frac{\log x}{\log\log x} < \log x$$

to bound the t's appearing inside the logs in (14), one gets

$$\log x < (1-\epsilon)\frac{\log x}{\log\log x}\Big(\log\log x + 2\log\log\log x\Big) = (1-\epsilon)\log x\Big(1 + \frac{2\log\log\log x}{\log\log x}\Big)$$

or, after some immediate simplifications,

$$\log\log x < \frac{2(1-\epsilon)}{\epsilon}\log\log\log x. \tag{15}$$

Since ϵ was fixed, it follows that inequality (15) cannot happen for arbitrarily large values of x. This proves that indeed (8) holds for any ϵ provided that x is large enough. We are now done. Indeed, going back to formula (7), it follows that

$$A_\delta(x) \geq 2^{t-1-\delta\log p_t}$$

or

$$B_\delta(x) = \log A_\delta(x) > \log 2(t - 1 - \delta\log p_t) > \log 2(t - 1 - \delta\log(2t\log t)), \tag{16}$$

where the last inequality in (16) follows from (11). By inequalities (8) and (16), we get

$$B_\delta(x) = t\log 2 - o(t) = \log 2(1-\epsilon)\frac{\log x}{\log\log x} - o(t).$$

Since ϵ could, in fact, be chosen arbitrarily small, we get

$$B_\delta(x) = \log 2\frac{\log x}{\log\log x} - o\Big(\frac{\log x}{\log\log x}\Big), \qquad 16)$$

which concludes the proof.

Remark.

We conjecture that the exact order of $B_\delta(x)$ is $\dfrac{\log x}{\log\log x} + O\Big(\dfrac{\log x}{(\log\log x)^2}\Big)$.

References

[1] F. Luca, "On the convergence of some series", preprint, 1998.

[2] J.B. Roser & L. Schoenfeld, "Approximate formulas for some functions of prime numbers", *Illinois J. of Math.* **6** (1962), pp. 64-94.

[3] S. Tabirca & T. Tabirca, "The convergence of Smarandache Harmonic Series", *SNJ* **9** (1998) No. 1-2, pp. 27-35.