ON THE DIVISOR PRODUCTS AND PROPER DIVISOR PRODUCTS SEQUENCES*

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ABSTRACT. Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n, $q_d(n)$ denotes the product of all proper divisors of n. In this paper, we study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the Makowski & Schinzel conjecture hold for the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$.

1. INTRODUCTION

Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n. That is, $p_d(n) = \prod_{d|n} d$. For example, $p_d(1) = 1$, $p_d(2) = 2$, $p_d(3) = 3$, $p_d(4) = 8$, $p_d(5) = 5$, $p_d(6) = 36$, $\cdots$, $p_d(p) = p$, $\cdots$. $q_d(n)$ denotes the product of all proper divisors of n. That is, $q_d(n) = \prod_{d|n, d<n} d$. For example, $q_d(1) = 1$, $q_d(2) = 1$, $q_d(3) = 1$, $q_d(4) = 2$, $q_d(5) = 1$, $q_d(6) = 6$, $\cdots$. In problem 25 and 26 of [1], Professor F.Smarandach asked us to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary methods to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the Makowski & Schinzel conjecture hold for $p_d(n)$ and $q_d(n)$. That is, we shall prove the following:

Theorem 1. *For any positive integer n, we have the inequality*

$$\sigma(\phi(p_d(n))) \geq \frac{1}{2} p_d(n),$$

where $\phi(k)$ is the Euler's function and $\sigma(k)$ is the divisor sum function.

Theorem 2. *For any positive integer n, we have the inequality*

$$\sigma(\phi(q_d(n))) \geq \frac{1}{2} q_d(n).$$

Key words and phrases. Makowski & Schinzel conjecture; Divisor and proper divisor product.
* This work is supported by the N.S.F. and the P.S.F. of P.R.China.

2. Some Lemmas

To complete the proof of the Theorems, we need the following two Lemmas:

Lemma 1. *For any positive integer n, we have the identities*

$$p_d(n) = n^{\frac{d(n)}{2}} \quad and \quad q_d(n) = n^{\frac{d(n)}{2}-1},$$

where $d(n) = \sum_{d|n} 1$ is the divisor function.

Proof. From the definition of $p_d(n)$ we know that

$$p_d(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So by this formula we have

$$p_d^2(n) = \prod_{d|n} n = n^{d(n)}. \tag{1}$$

From (1) we immediately get

$$p_d(n) = n^{\frac{d(n)}{2}}$$

and

$$q_d(n) = \prod_{d|n, d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}.$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $\alpha_i \geq 2$ $(i = 1, 2, \cdots, s)$, $p_j (j = 1, 2, \cdots, s)$ are some different primes with $p_1 < p_2 < \cdots < p_s$, then we have the estimate*

$$\sigma(\phi(n)) \geq \frac{6}{\pi^2} n.$$

Proof. From the properties of the Euler's function we have

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1}) \phi(p_2^{\alpha_2}) \cdots \phi(p_s^{\alpha_s}) \\ &= p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_s^{\alpha_s - 1} (p_1 - 1)(p_2 - 1) \cdots (p_s - 1). \end{aligned} \tag{2}$$

Let $(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$, where $\beta_i \geq 0$, $i = 1, 2, \cdots, s$, $r_j \geq 1$, $j = 1, 2, \cdots, t$ and $q_1 < q_2 < \cdots < q_t$ are different primes. Then from (2) we have

$$\begin{aligned}
\sigma(\phi(n)) &= \sigma(p_1^{\alpha_1+\beta_1-1}p_2^{\alpha_2+\beta_2-1}\cdots p_s^{\alpha_s+\beta_s-1}q_1^{r_1}q_2^{r_2}\cdots q_t^{r_t}) \\
&= \prod_{i=1}^{s}\frac{p_i^{\alpha_i+\beta_i}-1}{p_i-1}\prod_{j=1}^{t}\frac{q_j^{r_j+1}-1}{q_j-1} \\
&= p_1^{\alpha_1+\beta_1}p_2^{\alpha_2+\beta_2}\cdots p_s^{\alpha_s+\beta_s}q_1^{r_1}q_2^{r_2}\cdots q_t^{r_t}\prod_{i=1}^{s}\frac{1-\frac{1}{p_i^{\alpha_i+\beta_i}}}{p_i-1}\prod_{j=1}^{t}\frac{1-\frac{1}{q_j^{r_j+1}}}{1-\frac{1}{q_j}} \\
&= n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right)\prod_{j=1}^{t}\frac{1-\frac{1}{q_j^{r_j+1}}}{1-\frac{1}{q_j}} \\
&= n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right)\prod_{j=1}^{t}\left(1+\frac{1}{q_j}+\cdots+\frac{1}{q_j^{r_j}}\right) \\
&\geq n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right) \\
&\geq n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{2}}\right) \\
&\geq n\prod_{p}\left(1-\frac{1}{p^{2}}\right).
\end{aligned}$$

Noticing $\prod_{p}\frac{1}{1-\frac{1}{p^2}}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}$, we immediately get

$$\sigma(\phi(n))\geq n\cdot\prod_{p}\left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2}n.$$

This completes the proof of Lemma 2.

3. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 1. We separate n into prime and composite number two cases. If n is a prime, then $d(n)=2$. This time by Lemma 1 we have

$$p_d(n)=n^{\frac{d(n)}{2}}=n.$$

Hence, from this formula and $\phi(n)=n-1$ we immediately get

$$\sigma(\phi(p_d(n)))=\sigma(n-1)=\sum_{d|n-1}d\geq n-1\geq\frac{n}{2}=\frac{1}{2}p_d(n).$$

If n is a composite number, then $d(n) \geq 3$. If $d(n) = 3$, we have $n = p^2$, where p is a prime. So that

$$p_d(n) = n^{\frac{d(n)}{2}} = p^{d(n)} = p^3. \tag{3}$$

From Lemma 2 and (3) we can easily get the inequality

$$\sigma\left(\phi\left(p_d(n)\right)\right) = \sigma\left(\phi\left(p^3\right)\right) \geq \frac{6}{\pi^2} p^3 \geq \frac{1}{2} p_d(n).$$

If $d(n) \geq 4$, let $p_d(n) = n^{\frac{d(n)}{2}} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $p_1 < p_2 < \cdots < p_s$, then we have $\alpha_i \geq 2, i = 1, 2, \cdots, s$. So from Lemma 2 we immediately obtain the inequality

$$\sigma\left(\phi\left(p_d(n)\right)\right) \geq \frac{6}{\pi^2} p_d(n) \geq \frac{1}{2} p_d(n).$$

This completes the proof of Theorem 1.

The proof of Theorem 2. We also separate n into two cases. If n is a prime, then we have

$$q_d(n) = n^{\frac{d(n)}{2} - 1} = 1.$$

From this formula we have

$$\sigma\left(\phi\left(q_d(n)\right)\right) = 1 \geq \frac{1}{2} q_d(n).$$

If n is a composite number, we have $d(n) \geq 3$, then we discuss the following four cases. First, if $d(n) = 3$, then $n = p^2$, where p is a prime. So we have

$$q_d(n) = n^{\frac{d(n)}{2} - 1} = p^{d(n) - 2} = p.$$

From this formula and the proof of Theorem 1 we easily get

$$\sigma\left(\phi\left(q_d(n)\right)\right) \geq \frac{1}{2} q_d(n).$$

Second, if $d(n) = 4$, from Lemma 1 we may get

$$q_d(n) = n^{\frac{d(n)}{2} - 1} = n \tag{4}$$

and $n = p^3$ or $n = p_1 p_2$, where p, p_1 and p_2 are primes with $p_1 < p_2$. If $n = p^3$, from (4) and Lemma 2 we have

$$\begin{aligned} \sigma\left(\phi\left(q_d(n)\right)\right) &= \sigma\left(\phi(n)\right) = \sigma\left(\phi(p^3)\right) \\ &\geq \frac{1}{2} p^3 = \frac{1}{2} q_d(n). \end{aligned} \tag{5}$$

If $n = p_1 p_2$, we consider $p_1 = 2$ and $p_1 > 2$ two cases. If $2 = p_1 < p_2$, then $p_2 - 1$ is an even number. Supposing $p_2 - 1 = p_1^{\beta_1} p_2^{\beta_2} q_1^{r_1} \cdots q_t^{r_t}$ with $q_1 < q_2 < \cdots < q_t$,

$q_i (i = 1, 2, \cdots, t)$ are different primes and $r_j \geq 1$ $(j = 1, 2, \cdots, t)$, $\beta_1 \geq 1$, $\beta_2 \geq 0$. Note that the proof of Lemma 2 and (4) we can obtain

$$\begin{aligned}
\sigma\left(\phi\left(q_d(n)\right)\right) &= \sigma\left(\phi(n)\right) \\
&= n \prod_{i=1}^{2}\left(1 - \frac{1}{p_i^{1+\beta_i}}\right) \prod_{j=1}^{t}\left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{r_j}}\right) \\
&\geq n\left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2}\right) \\
&\geq n(1 - \frac{1}{4})(1 - \frac{1}{3}) \\
&= \frac{1}{2} q_d(n). \qquad (6)
\end{aligned}$$

If $2 < p_1 < p_2$, then both $p_1 - 1$ and $p_2 - 1$ are even numbers. Let $(p_1 - 1)(p_2 - 1) = p_1^{\beta_1} p_2^{\beta_2} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$ with $q_1 < q_2 < \cdots < q_t, q_i (i = 1, 2, \cdots, t)$ are different primes and $r_j \geq 1 (j = 1, 2, \cdots, t), \beta_1, \beta_2 \geq 0$, then we have $q_1 = 2$ and $r_1 \geq 2$. So from the proof of Lemma 2 and (4) we have

$$\begin{aligned}
\sigma\left(\phi\left(q_d(n)\right)\right) &= \sigma\left(\phi(n)\right) \\
&= n \prod_{i=1}^{2}\left(1 - \frac{1}{p_i^{1+\beta_i}}\right) \prod_{j=1}^{t}\left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{r_j}}\right) \\
&\geq n \prod_{i=1}^{2}\left(1 - \frac{1}{p_i}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) \\
&\geq n \prod_{i=1}^{2}\left(1 - \frac{1}{p_i}\right)(1 + \frac{1}{3})(1 + \frac{1}{5}) \\
&\geq n \prod_{i=1}^{2}\left[(1 - \frac{1}{p_i})(1 + \frac{1}{p_i})\right] \\
&\geq n \prod_{p}\left(1 - \frac{1}{p^2}\right) \\
&\geq n \frac{6}{\pi^2} \\
&\geq \frac{1}{2} q_d(n). \qquad (7)
\end{aligned}$$

Combining (5), (6) and (7) we obtain

$$\sigma\left(\phi\left(q_d(n)\right)\right) \geq \frac{1}{2} q_d(n) \quad \text{if} \quad d(n) = 4.$$

Third, if $d(n) = 5$, we have $n = p^4$, where p is a prime. Then from Lemma 1 and Lemma 2 we immediately get

$$\sigma\left(\phi\left(q_d(n)\right)\right) = \sigma\left(\phi\left(p^6\right)\right) \geq \frac{6}{\pi^2} p^6 = \frac{1}{2} q_d(n).$$

Finaly, if $d(n) \geq 6$, then from Lemma 1 and Lemma 2 we can easily obtain

$$\sigma(\phi(q_d(n))) \geq \frac{1}{2} q_d(n).$$

This completes the proof of Theorem 2.

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