ON THE DIVISOR PRODUCT SEQUENCES

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ABSTRACT. The main purpose of this paper is to study the asymptotic property of the divisor product sequences, and obtain two interesting asymptotic formulas.

1. INTRODUCTION AND RESULTS

A natural number a is called a divisor product of n if it is the product of all positive divisors of n. We write it as $P_d(n)$, it is easily to prove that $P_d(n) = n^{\frac{d(n)}{2}}$, where $d(n)$ is the divisor function. We can also define the proper divisor product of n as the product of all positive divisors of n but n, we denote it by $p_d(n)$, and $p_d(n) = n^{\frac{d(n)}{2}-1}$. It is clear that the $P_d(n)$ sequences is

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, \cdots;$$

The $p_d(n)$ sequences is

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 1, 400, 21, \cdots.$$

In reference [1], Professor F. Smarandache asked us to study the properties of these two sequences. About these problems, it seems that none had studied them before. In this paper, we use the analytic methods to study the asymptotic properties of these sequences, and obtain two interesting asymptotic formulas. That is, we shall prove the following two Theorems.

Theorem 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{1}{P_d(n)} = \ln\ln x + C_1 + O(\frac{1}{\ln x}).$$

where C_1 is a constant.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{1}{p_d(n)} = \pi(x) + (\ln\ln x)^2 + B\ln\ln x + C_2 + O(\frac{\ln\ln x}{\ln x}).$$

where $\pi(x)$ is the number of all primes $\leq x$, B and C_2 are constants.

Key words and phrases. Divisor products of n; Proper divisor products of n ; Asymptotic formula..

2. Some lemmas

To complete the proof of the theorems, we need following several lemmas.

Lemma 1. *For any real number $x \geq 2$, there is a constant A such that*

$$\sum_{p\leq x}\frac{1}{p}=\ln\ln x+A+O(\frac{1}{\ln x}).$$

Proof. See Theorem 4.12 of reference [2].

Lemma 2. *Let $x \geq 2$, then we have*

$$\sum_{p\leq x}\frac{\ln p}{p}=\ln x+C+O(\frac{1}{\ln x}).$$

where C is constant.

Proof. See reference [4].

Lemma 3. *Let $x \geq 4$, p and q are primes. Then we have the asymptotic formula*

$$\sum_{pq\leq x}\frac{1}{pq}=(\ln\ln x)^2+A\ln\ln x+C_3+O(\frac{\ln\ln x}{\ln x}),$$

where A and C_3 are constants.

Proof. From Lemma 1 and Lemma 2 we have

$$\begin{aligned}\sum_{pq\leq x}\frac{1}{pq}&=2\sum_{p\leq\sqrt{x}}\frac{1}{p}\sum_{q\leq\frac{x}{p}}\frac{1}{q}-\left(\sum_{p\leq\sqrt{x}}\frac{1}{p}\right)^2\\
&=2\sum_{p\leq\sqrt{x}}\frac{1}{p}\left(\ln\ln x+\ln(1-\frac{\ln p}{\ln x})+A+O(\frac{1}{\ln x})\right)\\
&\quad-\left(\ln\ln x+A-\ln 2+O(\frac{1}{\ln x})\right)^2\\
&=2\sum_{p\leq\sqrt{x}}\frac{1}{p}\left(\ln\ln x-\left(\frac{\ln p}{\ln x}+\frac{1}{2}(\frac{\ln p}{\ln x})^2+\frac{1}{3}(\frac{\ln p}{\ln x})^3+\cdots\frac{1}{n}(\frac{\ln p}{\ln x})^n+\cdots\right)\right)\\
&\quad+2A\sum_{p\leq\sqrt{x}}\frac{1}{p}+O(\frac{\ln\ln x}{\ln x})-\left(\ln\ln x+A-\ln 2+O(\frac{1}{\ln x})\right)^2\\
&=(\ln\ln x)^2+2A\ln\ln x+C_3+O(\frac{\ln\ln x}{\ln x}).\end{aligned}$$

This proves Lemma 3.

3. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 2. Note that the definition of $p_d(n)$, we can separate n into four parts according to $d(n) = 2, 3, 4$ or $d(n) \geq 5$.

$$d(n) = \begin{cases} 2, & \text{if } n = p, \ p_d(n) = 1; \\ 3, & \text{if } n = p^2, \ p_d(n) = p; \\ 4, & \text{if n=p}_i p_j \text{ or } n = p^3; \ p_d(n) = p_i p_j \text{ or } p^3; \\ \geq 5, & \text{others, p}_d(n) = n^{\frac{d(n)}{2}-1}. \end{cases}$$

Then by Lemma 1, 2 and 3 we have

$$\begin{aligned} \sum_{n \leq x} \frac{1}{p_d(n)} &= \sum_{p \leq x} 1 + \sum_{p_i p_j \leq x} \frac{1}{p_i p_j} + \sum_{p^2 \leq x} \frac{1}{p} + \sum_{p^3 \leq x} \frac{1}{p^3} + \sum_{n \leq x, d(n) \geq 5} \frac{1}{n^{\frac{d(n)}{2}-1}} \\ &= \pi(x) + (\ln \ln x)^2 + 2A \ln \ln x + C_3 + O(\frac{\ln \ln x}{\ln x}) + \ln \ln x + A - \\ &\qquad \ln 2 + O(\frac{1}{\ln x}) + C_4 + O\left(\frac{1}{x^{\frac{2}{3}}}\right) + C_5 + O\left(\frac{1}{\sqrt{x}}\right) \\ &= \pi(x) + (\ln \ln x)^2 + B \ln \ln x + C_2 + O(\frac{\ln \ln x}{\ln x}). \end{aligned}$$

This completes the proof of Theorem 2.

Similarly, we can also prove Theorem 1. Note that the definition of $P_d(n)$, we have

$$\begin{aligned} \sum_{n \leq x} \frac{1}{P_d(n)} &= \sum_{p \leq x} \frac{1}{p} + \sum_{p_i p_j \leq x} \frac{1}{(p_i p_j)^2} + \sum_{p^2 \leq x} \frac{1}{p^3} + \sum_{p^3 \leq x} \frac{1}{p^6} + \sum_{n \leq x, d(n) \geq 5} \frac{1}{n^{\frac{d(n)}{2}}} \\ &= \ln \ln x + C_1 + O(\frac{1}{\ln x}). \end{aligned}$$

This completes the proof of Theorem 1.

References

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