ON THE FUNCTIONAL EQUATION $(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n$

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Abstract For any positive integer n, let $S(n)$ be the Smarandache function of n. Let r be a fixed positive integer with $r \geqslant 3$. In this paper we give a necessary and sufficient condition for the functional equation $(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n$ to have positive integer solutions n.

Key words Smarandache function, functional equation, solvability.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, let the arithmetic function

(1) $S(n) = \min\{a \mid a \in \mathbb{N},\ n \mid a!\}$

Then $S(n)$ is called the Smarandache function of n For a fixed $r \in \mathbb{N}$ with $r \geqslant 3$, we discuss the solvability of the functional equation

(2) $(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n, n \in \mathbb{N}$

There are many unsolved questions concerned this equation(see [1]). A computer search showed that if $r = 3$, then (2) has no solution n with $n \leqslant 10000$. In this paper we prove a general result as follows.

Theorem For any fixed $r \in$ ⁚ with $r \geqslant 3$, a positive integer n is a solution of (2) if and only if $n = p(p^{r-1} + p^{r-2} + \cdots + 1)$, where p is an odd prime satisfying $p^{r-1} + p^{r-2} + \cdots + 1 \mid (p-1)!$.

By our theorem, we find that if $r = 3$, then (2) has exactly two solutions $n = 305319$ and $n = 499359$ with $n < 1000000$.

2 Preliminaries

Lemma 1 For any u, $v \in \mathbb{N}$ with $\gcd(u, v) = 1$, we have $S(uv) = \max(S(u), S(v))$.

Proof Let $a = S(u)$, $b = S(v)$ and $c = S(uv)$. By (1), a, b, c are least positive integers satisfying

(3) $u \mid a!$, $v \mid b!$, $uv \mid c!$,

respectively. We see from (3) that

(4) $c \geqslant \max(a, b)$

If $a \geqslant b$, then $u \mid a!$ and $v \mid a!$ by (3). Since $\gcd(u, v) = 1$, we get $uv \mid a!$. It implies that $a \geqslant c$. Therefore, by (4), we obtain $c = a = \max(a, b)$. By the same method, we can prove that if $a \leqslant b$, then $c = b = \max(a, b)$. The lemma is proved.

Lemma 2 If $S(u) = u$, then $u = 1, 4$ or p, where p is a prime.

Proof See [3].

Lemma 3 If $u > 1$, where $u \in \mathbb{N}$, then u has a prime factor p such that $p \mid S(u)$.

Proof Let $u = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the factorization of u. It is a well known fact that $S(u) = \max(S(p_1^{r_1}), S(p_2^{r_2}), \cdots, S(p_k^{r_k}))$ and $p_i \mid S(p_i^{r_i})$ for $i = 1, 2, \cdots, k$ (see [2]). The lemma follows immediately.

3 Proof of Theorem

Let $n=p(p^{r-1}+p^{r-2}+\cdots+1)$, where p is an odd prime satisfying $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$. Then, by (1), we get $S(n)=p$. Therefore, n is a solution of (2).

On the other hand, let n be a solution of (2). Then we have $n>1$. Further, let $t=S(n)$. We get from (2) that

(5) $\quad t(t^{r-1}+t^{r-2}+\cdots+1)=n.$

Since $\gcd(t, t^{r-1}+t^{r-2}+\cdots+1)=1$, by Lemma 1, we see from (5) that

(6) $t=S(n)=S(t(t^{r-1}+t^{r-2}+\cdots+1))$

$\quad =\max(S(t), S(t^{r-1}+t^{r-2}+\cdots+1))$

If $S(t)\leqslant S(t^{r-1}+t^{r-2}+\cdots+1)$, then from (6) we get

(7) $\quad t=S(t^{r-1}+t^{r-2}+\cdots+1)$

Since $t^{r-1}+t^{r-2}+\cdots+1>1$, by Lemma 3, $t^{r-1}+t^{r-2}+\cdots+1$ has a prime factor p such that $p \mid S(t^{r-1}+t^{r-2}+\cdots+1)$. Hence, by (7), we get $p \mid t$. However, since $\gcd(t, t^{r-1}+t^{r-2}+\cdots+1)=1$, it is impossible. So we have

(8) $\quad S(t)>S(t^{r-1}+t^{r-2}+\cdots+1)$

and

(9) $\quad t=S(t)$,

by (6)

On applying Lemma 2, we see from (9) that either $t=4$ or $t=p$, where p is a prime. If $t=4$, then $n=4$, 8, 12 or 24. However, since $r\geqslant 3$, we get from (5) that $t^r+t^{r-1}+\cdots+t\geqslant 4^3+4^2+4>24\geqslant n$, a contradiction. If $t=p$, then from (8) and (9) we obtain

(10) $\quad S(p^{r-1}+p^{r-2}+\cdots+1)<S(t)=S(p)=p$

It implies that $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$ and $p>2$. Therefore, we see from (5) that if n is a solution of (2), then $n=p(p^{r-1}+p^{r-2}+\cdots+1)$, where p is an odd prime satisfying $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$. Thus, the theorem is proved.

References

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