On the Functional Equation $S(n)^2 + S(n) = kn$

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Abstract

For any positive integer a, let S(a) denote the Smarandache function of a. In this paper, we prove that the equation $S(n)^2 + S(n) = kn$ has infinitely many positive integer solutions for every positive integer k. Moreover, the size of the number of solutions does not depend on the parity of k.

Key Words: Smarandache function, functional equation, number of solutions.

1. Introduction

Let N be the set of positive integers. For any positive integer a, let

(1) $S(a) = \min \{ r \mid r \in N, a \mid r! \}$.

Then S(a) is called the Smarandache function of a. Let k be a fixed positive integer. In this paper we deal with the equation

(2) $S(n)^2 + S(n) = kn, n \in N$.

For any positive integer x, let N(k,x) denote the number of solutions n with $n \leq x$, and let N(k) denote the number of all solutions n of (2). A computer search showed that $N(1, 10^4) = 23$, $N(2, 10^4) = 33$, $N(3, 10^4) = 20$, $N(4,10^4) = 24$, $N(5,10^4) = 11$ and $N(6, 10^4) = 26$. In [1] Ashbacher posed the following questions:

Question 1: Is $N(k) = \infty$ for k = 1, 2, 3, 4, 5 or 6?
Question 2: Is there a positive integer k for which N(k) = 0?
Question 3: Is there a largest positive integer for which N(k) > 0?
Question 4: Is there more solutions n when k is even than when k is odd?

In this paper, we completely solve the above-mentioned questions. In fact, we prove a general result as follows:

Theorem: The positive integer n is a solution of (2) if and only one of the following conditions is satisfied.

(i) n = 1 for k = 2.
(ii) n = 4 for k = 5.
(iii) n = p(p+1) for k = 1, where p is a prime with p > 3.
(iv) n = p(p+1)/k for k > 1, where p is a prime with $p \equiv -1 \pmod{k}$.

Corollary 1: As $x \to \infty$, we have

$$N(k,x) \sim 2\sqrt{(kx)} / (\varphi(k)\log(kx)) .$$

Corollary 2: For any positive integers k_1 and k_2, we have

$$\frac{N(k_1)}{N(k_2)} = \frac{\varphi(k_2)}{\varphi(k_1)} \sqrt{(k_1/k_2)}$$

By our results, we observe that (2) has infinitely many solutions n for every positive integer k. Moreover, the size of N(k,x) does not depend on the parity of k.

2. Preliminaries

Lemma 1: For any positive integers u and v, we have $S(u) \le S(uv)$.

Proof: Let $a = S(u)$ and $b = S(uv)$. By (1), a and b are least positive integers satisfying $u \mid a!$ and $uv \mid b!$ respectively. So we have $a \le b$. The lemma is proved.

Lemma 2: For any positive integer u with $u > 1$, there exists a prime factor d such that $d \mid S(u)$.

Proof: Let $u = p_1^{t1} p_2^{t2} \dots p_k^{tk}$ be the prime factorization of u. Then, by [2], we have

$$S(u) = \max\left(S(p_1^{t1}), S(p_2^{t2}), \dots, S(p_k^{tk}) \right)$$

and $p_i \mid S(p_i^{ti})$ for $i = 1, 2, 3, \dots, k$. This proves the lemma.

Lemma 3: For any positive integer u, we have

$$S(u) \begin{cases} = u, & \text{if } u = 1, 4 \text{ or } p, \text{ where p is a prime.} \\ \le u/2, & \text{otherwise.} \end{cases}$$

Proof: See [4].

Lemma 4: For any coprime positive integers, u and v, we have $S(uv) = \max(S(u), S(v))$.

Proof: Let $a = S(u)$, $b = S(v)$ and $c = S(uv)$. By (1), a, b and c are least positive integers satisfying $u \mid a!$, $v \mid b!$ and $uv \mid c!$ respectively. This implies that $c \ge \max(a,b)$.
If $a \ge b$, then we have $u \mid a!$ and $v \mid a!$. Since $\gcd(u,v) = 1$, we get $uv \mid a!$. So we have $a \ge c$. This implies that $c = a = \max(a,b)$. By the same method, we can prove that if $a \le b$, then $c = b = \max(a,b)$. The lemma is proved.

Lemma 5: For any positive number x, let $\Pi(x)$ denote the number of primes p with $p \le x$. As $x \to \infty$, we have $\Pi(x) \sim x/\log x$.

Proof: See [3].

Lemma 6: Let a,b be integers satisfying $a > 1$ and $\gcd(a,b) = 1$. For any positive number x, let $\Pi(x;a,b)$ denote the number of primes p such that $p \le x$ and $p \equiv b \pmod a$. As $x \to \infty$, we have $\Pi(x;a,b) \sim x/\varphi(a)\log x$, where $\varphi(a)$ is the Euler function of a.

Proof: See [5].

3. Proofs

Proof of Theorem: Clearly, if n satisfy (i) or (ii), then it is a solution of (2). If n satisfy (iii), then $n = p(p+1)$, where p is a prime with $p > 3$. Since $\gcd(p,p+1) = 1$, by Lemma 4, we get

(5) $$S(n) = S(p(p+1)) = \max(S(p), S(p+1)).$$

Further, since $p+1 \geq 6$ is not a prime, by Lemma 3, we get $S(p+1) \leq (p+1)/2 < p$. Hence, we see from (5) that $S(n) = S(p) = p$. It implies that $S(n)^2 + S(n) = p^2 + p = n$ and n is a solution of (2) for $k = 1$. By the same method, we can prove that if n satisfy the condition (iv), then it is a solution of (2) for $k > 1$. Thus, the sufficient condition of our theorem is proved.

We now prove the necessary condition. Let n be a solution of (2), and let $t = S(n)$. We get from (2) that

(6) $$t(t+1) = kn.$$

If $n = 1$ or 4, then $t = 1$ or 4, and n is a solution of (2) for $k = 2$ or 5. From below, we may assume that $n \neq 1$ or 4. Since $\gcd(t,t+1) = 1$, by Lemma 4, we get from (6) that

(7) $$S(kn) = S(t(t+1)) = \max(S(t),S(t+1)).$$

If $S(t) \leq S(t+1)$, then from (7) we get

(8) $$S(kn) = S(t+1).$$

By Lemma 1, we have $S(kn) \geq S(n) = t$. Hence, by (8) we obtain

(9) $$S(t+1) \geq t.$$

Since $n \neq 1$ or 4, by Lemma 3, we see from (9) that either $t = 3$ or $t = p-1$, where p is a prime. When $t = 3$, we get $n = 3$ or 6. Then n satisfies the condition (iv). When $t = p-1$, we have $S(n) = p-1$ and

(10) $$S(kn) = p,$$

by (8). Since p is a prime, by Lemma 2, we see from (10) that $p \mid kn$. If $p \mid k$, then k/p is a positive integer and $t = p - 1 = kn/p$ by (6). However, by Lemmas 1 and 3, it implies that

$$p - 1 > S(p - 1) = S(kn/p) \geq S(n) = t = p - 1, \text{ a contradiction.}$$

If $S(t) > S(t+1)$, then from (17) we get

(11) $$S(kn) = S(t).$$

Since $S(kn) \geq S(n) = t$, by Lemmas 1 and 3, we see from (11) that $S(t) = t$. Since $n \neq 1$ or 4, by Lemma 4, we get $t = p$, where p is a prime. Hence, by (6), we obtain

(12) $$p(p+1) = kn.$$

Further, since $S(n) = p$, by Lemma 2, we have $p \mid n$ and n/p is a positive integer. Then, by (12) we get $p \equiv -1 \pmod{k}$. Furthermore, since $n \neq 4$, we get from (12) that $p > 3$, for $k = 1$. This implies that n satisfies the condition (iii) of (iv). Thus, the theorem is proved.

Proof of Corollaries 1 and 2. Let $\Pi(x)$ and $\Pi(x;a,b)$ be defined as in Lemmas 5 and 6 respectively. By Theorem, we have

$$N(k,x) = \begin{cases} \Pi\left(\sqrt{(x + 1/4)} - 1/2\right) - 2, & \text{if } k = 1, \\ \Pi\left(\sqrt{(2x + 1/4)} - 1/2\right), & \text{if } k = 2, \\ \Pi\left(\sqrt{(5x + 1/4)} - 1/2; 5,-1\right) + 1, & \text{if } k = 5, \\ \Pi\left(\sqrt{(kx + 1/4)} - 1/2; k,-1\right), & \text{otherwise.} \end{cases}$$

Therefore, by Lemmas 5 and 6, we get the corollaries immediately.

References

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