On The Functional Equation $Z(n) + \varphi(n) = d(n)$

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Abstract: For any positive integer n, let $d(n)$, $\varphi(n)$ and $Z(n)$ denote the divisor function, the Euler function and the pseudo-Smarandache function of n respectively. In this paper, we prove that the functional equation $Z(n) + \varphi(n) = d(n)$ has no solution n.

Key words: divisor function, Euler function, pseudo-Smarandache function.

Let N be the set of all positive integers. For any positive integer n, let

$$d(n) = \sum_{d|n} 1, \tag{1}$$

$$\varphi(n) = \sum_{\substack{1 \le m \le n \\ \gcd(m,n)=1}} 1, \tag{2}$$

$$Z(n) = \min \left\{ a \mid a \in N,\ n \mid \sum_{j=1}^{a} j \right\}. \tag{3}$$

Then $d(n)$, $\varphi(n)$ and $Z(n)$ are called the divisor function, the Euler function and the Pseudo-Smarandache function of n respectively. In [1], Ashbacher posed the following unsolved question:

Question: How many solutions n are there to the functional equation

$$Z(n) + \varphi(n) = d(n),\ n \in N\ ? \tag{4}$$

In this paper, we completely solve the above-mentioned question as follows:

Theorem: The equation $Z(n) + \varphi(n) = d(n)$, $n \in N$ has no solution.

Proof: Let n be a solution of (4). A computer search showed that (4) has no solution with $n \le 10000$ (see [1]). So we have $n > 10000$. Let

$$n = p_1^{r1} p_2^{r2} \ldots p_k^{rk} \tag{5}$$

be the prime factorization of n. By [2, theorems 62 and 273], we see from (1), (2) and (5) that

$$d(n) = (r_1 + 1)(r_2 + 1) \ldots (r_k + 1) \tag{6}$$

$$\varphi(n) = n \prod_{i=1}^{k} (1 - 1/p_i) \tag{7}$$

On the other hand, it is a well-known fact that

(8) $n \mid \frac{1}{2} Z(n)(Z(n)+1)$

(see [1]). From (8) we get

$$Z(n) \geq \sqrt{(2n + \frac{1}{4}} - \frac{1}{2}) .$$

Therefore, by (4), (5), (6), (7) and (9), we obtain

(10) $1 \geq f(n) + g(n)$

where

(11) $$f(n) = \prod_{i=1}^{k} (1 - 1/p_i)(p^{r_i}/(r_i+1)),$$

(12) $$g(n) = \sqrt{2} \prod_{i=1}^{k} (p_i^{r_i/2}/(r_i+1)) - \frac{1}{2} \prod_{i=1}^{k} 1/(r_i + 1) .$$

Clearly, we see from (12) that $g(n) > 0$ for any positive integer n with $n > 1$. Hence, we get from (10) that

(13) $f(n) < 1$.

If $k = 1$, then $n = p_1^{r_1}$ and $Z(n) \geq p_1^{r_1} - 1$ by (3). Hence, by (1) and (2), n is not a solution of (4). This implies that $k \geq 2$.
If $k \geq 3$, then $p_k \geq 5$ and $f(n) \geq 1$, by (11). This contradicts with (13). So we have $k = 2$. Then (11) can be written as

(14) $$f(n) = (1 - 1/p_1)(1 - 1/p_2)\left((p_1^{r_1} p_2^{r_2})/((r_1+1)(r_2+1))\right) .$$

If $p_2 > 3$, then from (14) we get $f(n) \geq 1$, a contradiction. Hence, we deduce that $p_1 = 2$ and $p_2 = 3$. Then, by (13) and (14), we obtain

(15) $$f(n) = (2^{r_1} 3^{r_2})/(3(r_1+1)(r_2+1)) < 1.$$

From (15), we can calculate that $(r_1, r_2) = (1,1)$ or $(2,1)$. This implies that $n \leq 12$, a contradiction. Thus, (4) has no solution n. The theorem is proved.

References

(1) C. Ashbacher, 'The Pseudo-Smarandache Function and the Classical Functions of Number Theory", **Smarandache Notions J.**, 9(1995), 78-81.

(2) G. H. Hardy and E. M. Wright, **An Introduction to the Theory of Numbers**, Oxford University Press, 1937.