

# ON THE INFERIOR AND SUPERIOR $k$ -TH POWER PART OF A POSITIVE INTEGER AND DIVISOR FUNCTION

ZHENG JIANFENG

Shaanxi Financial & Economics Professional College,  
Xianyang, Shaanxi, P.R.China

**ABSTRACT:** For any positive integer  $n$ , let  $a(n)$  and  $b(n)$  denote the inferior and superior  $k$ -th power part of  $n$  respectively. That is,  $a(n)$  denotes the largest  $k$ -th power less than or equal to  $n$ , and  $b(n)$  denotes the smallest  $k$ -th power greater than or equal to  $n$ . In this paper, we study the properties of the sequences  $\{a(n)\}$  and  $\{b(n)\}$ , and give two interesting asymptotic formulas.

**Key words and phrases:** Inferior and superior  $k$ -th power part; Mean value; Asymptotic formula.

## 1. INTRODUCTION

For a fixed positive integer  $k > 1$ , and any positive integer  $n$ , let  $a(n)$  and  $b(n)$  denote the inferior and superior  $k$ -th power part of  $n$  respectively. That is,  $a(n)$  denotes the largest  $k$ -th power less than or equal to  $n$ ,  $b(n)$  denotes the smallest  $k$ -th power greater than or equal to  $n$ . For example, let  $k=2$  then  $a(1)=a(2)=a(3)=1, a(4)=a(5)=\dots=a(7)=4, \dots, b(1)=1, b(2)=b(3)=b(4)=4, b(5)=b(6)=\dots=b(8)=8 \dots$ ; let  $k=3$  then  $a(1)=a(2)=\dots=a(7)=1, a(8)=a(9)=\dots=a(26)=8, \dots, b(1)=1, b(2)=b(3)=\dots=b(8)=8, b(9)=b(10)=\dots=b(27)=27 \dots$ . In problem 40 and 41 of [1], Professor F. Smarandache asks us to study the properties of the sequences  $\{a(n)\}$  and  $\{b(n)\}$ . About these problems, Professor Zhang Wenpeng [4] gave two interesting asymptotic formulas of the cure part of a positive integer. In this paper, we give asymptotic formulas of the  $k$ -th power part of a positive integer. That is, we shall prove the following:

**Theorem 1.** For any real number  $x > 1$ , we have the asymptotic formula

$$\sum_{n \leq x} d(a(n)) = \frac{1}{kk!} \left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x \ln^k x + A_1 x \ln^{k-1} x + \dots + A_{k-1} x \ln x + A_k x + O(x^{1-\frac{1}{2k}+\varepsilon}),$$

where  $A_0, A_1, \dots, A_k$  are constants, especially when  $k$  equals to 2,  $A_0=1$ ;  $d(n)$  denotes the Dirichlet divisor function,  $\varepsilon$  is any fixed positive number.

For the sequence  $\{b(n)\}$ , we can also get similar result.

**Theorem 2.** For any real number  $x > 1$ , we have the asymptotic formula

$$\sum_{n \leq x} d(b(n)) = \frac{1}{kk!} \left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x \ln^k x + A_1 x \ln^{k-1} x + \dots + A_{k-1} x \ln x + A_k x + O(x^{1-\frac{1}{2k}+\varepsilon})$$

## 2. A SIMPLE LEMMA

To complete the proof of the theorems, we need following

**Lemma 1.** For any real number  $x > 1$ , we have the asymptotic formula

$$\sum_{n \leq x} d(n^k) = \frac{1}{k!} \left(\frac{6}{\pi^2}\right)^{k-1} B_0 x \ln^k x + B_1 x \ln^{k-1} x + \cdots + B_{k-1} x \ln x + B_k x + O(x^{\frac{1}{2}+\varepsilon}).$$

where  $B_0, B_1, \dots, B_k$  are constants, especially when  $k=2, A_0=1$ ;  $\varepsilon$  is any fixed positive number.

Proof. Let  $s = \sigma + it$  be a complex number and  $f(s) = \sum_{n=1}^{\infty} \frac{d(n^k)}{n^s}$ .

Note that  $d(n^k) \ll n^\varepsilon$ , So it is clear that  $f(s)$  is a Dirichlet series absolutely convergent in  $\text{Re}(s) > 1$ , by the Euler Product formula [2] and the definition of  $d(n)$  we have

$$\begin{aligned} f(s) &= \prod_p \left( 1 + \frac{d(p^k)}{p^s} + \frac{d(p^{2k})}{p^{2s}} + \cdots + \frac{d(p^{kn})}{p^{ns}} + \cdots \right) \\ &= \prod_p \left( 1 + \frac{k+1}{p^s} + \frac{2k+1}{p^{2s}} + \cdots + \frac{kn+1}{p^{ns}} + \cdots \right) \\ &= \zeta^2(s) \prod_p \left( 1 + (k-1) \frac{1}{p^s} \right) \\ &= \zeta^2(s) \prod_p \left( \left(1 + \frac{1}{p^s}\right)^{k-1} - C_{k-1}^2 \frac{1}{p^{2s}} - \cdots - \frac{1}{p^{(k-1)s}} \right) \\ &= \frac{\zeta^{k+1}(s)}{\zeta^{k-1}(2s)} g(s). \end{aligned} \tag{1}$$

where  $\zeta(s)$  is Riemann zeta-function and  $\prod_p$  denotes the product over all primes.

From (1) and Perron's formula [3] we have

$$\sum_{n \leq x} d(n^k) = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{\zeta^{k+1}(s)}{\zeta^{k-1}(2s)} g(s) \frac{x^s}{s} ds + O\left(\frac{x^{2+\varepsilon}}{T}\right), \tag{2}$$

where  $g(s)$  is absolutely convergent in  $\text{Re}(s) > \frac{1}{2} + \varepsilon$ . We move the integration in (2) to

$\text{Re}(s) = \frac{1}{2} + \varepsilon$ . The pole at  $s=1$  contributes to

$$\frac{1}{k!} \left(\frac{6}{\pi^2}\right)^{k-1} B_0 x \ln^k x + B_1 x \ln^{k-1} x + \cdots + B_{k-1} x \ln x + B_k x, \tag{3}$$

where  $B_0, B_1, \dots, B_k$  are constants, especially when  $k=2, B_0=1$ .

For  $\frac{1}{2} \leq \sigma < 1$ , note that  $\zeta(s) = \zeta(\sigma + it) \leq |t|^{\frac{1-\sigma}{2}+\varepsilon}$ . Thus, the horizontal integral contributes to

$$O\left(x^{\frac{1}{2}+\varepsilon} + \frac{x^2}{T}\right), \quad (4)$$

and the vertical integral contributes to

$$O\left(x^{\frac{1}{2}+\varepsilon} \ln^4 T\right). \quad (5)$$

On the line  $\operatorname{Re}(s) = \frac{1}{2} + \varepsilon$ , taking parameter  $T = x^{\frac{3}{2}}$ , then combining (2), (3), (4) and (5) we

have

$$\sum_{n \leq x} d(n^k) = \frac{1}{k!} \left(\frac{6}{\pi^2}\right)^{k-1} B_0 x \ln^k x + B_1 x \ln^{k-1} x + \cdots + B_k x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This proves Lemma 1.

### 3. PROOFS OF THE THEOREMS

Now we complete the proof of the Theorems. First we prove Theorem 1.

For any real number  $x > 1$ , Let  $M$  be a fixed positive integer such that

$$M^k \leq x < (M+1)^k, \quad (6)$$

then, from the definition of  $a(n)$ , we have

$$\begin{aligned} \sum_{n \leq x} d(a(n)) &= \sum_{m=2}^M \sum_{(m-1)^k \leq n < m^k} d(a(n)) + \sum_{M^k \leq n \leq x} d(a(n)) \\ &= \sum_{m=1}^{M-1} \sum_{m^k \leq n < (m+1)^k} d(m^k) + \sum_{M^k \leq n \leq x} d(M^k) \\ &= \sum_{m=1}^{M-1} (C_k^1 m^{k-1} + C_k^2 m^{k-2} + \cdots + 1) d(m^k) + O\left(\sum_{M^k \leq n \leq (M+1)^k} d(M^k)\right), \\ &= k \sum_{m=1}^M m^{k-1} d(m^k) + O(M^{k-1+\varepsilon}), \end{aligned} \quad (7)$$

where we have used the estimate  $d(n) \ll n^\varepsilon$ .

Let  $B(y) = \sum_{n \leq y} d(n^k)$ , then by Abel's identity and Lemma 1, we have

$$\begin{aligned} \sum_{m=1}^M m^{k-1} d(m^k) &= M^{k-1} B(M) - (k-1) \int_1^M y^{k-2} B(y) dy + O(1) \\ &= M^{k-1} \left( \frac{1}{k!} \left(\frac{6}{\pi^2}\right)^{k-1} B_0 M \ln^k M + B_1 M \ln^{k-1} M + \cdots + B_k M \right) \end{aligned}$$

$$\begin{aligned}
& - (k-1) \int_1^M \left( \frac{1}{k!} \left( \frac{6}{\pi^2} \right)^{k-1} B_0 y^{k-1} \ln^k y + B_1 y^{k-1} \ln^{k-1} y + \dots + B_k y^{k-1} \right) dy \\
& + O \left( M^{k-\frac{1}{2}+\varepsilon} \right) \\
& = \frac{1}{kk!} \left( \frac{6}{\pi^2} \right)^{k-1} B_0 M^k \ln^k M + C_1 M^k \ln^{k-1} M + \dots + C_{k-1} M^k + O \left( M^{k-\frac{1}{2}+\varepsilon} \right). \quad (8)
\end{aligned}$$

Applying (7) and (8) we obtain the asymptotic formula

$$\sum_{n \leq x} d(a(n)) = \frac{1}{kk!} \left( \frac{6}{\pi^2} \right)^{k-1} B_0 M^k \ln^k M + C_1 M^k \ln^{k-1} M + \dots + C_{k-1} M^k + O \left( M^{k-\frac{1}{2}+\varepsilon} \right), \quad (9)$$

where  $B_0, C_1, \dots, C_{k-1}$  are constants.

From (6) we have the estimates

$$\begin{aligned}
0 \leq x - M^k & < (M+1)^k - M^k = kM^{k-1} + C_k^2 M^{k-2} + \dots + 1 \\
& = M^{k-1} \left( k + C_k^2 \frac{1}{M} + \dots + \frac{1}{M^{k-1}} \right) \ll x^{\frac{k-1}{k}}, \quad (10)
\end{aligned}$$

and

$$\ln^k x = k^k \ln^k M + O \left( \frac{\ln^{k-1} x}{x^{\frac{1}{k}}} \right) = k^k \ln^k M + O \left( x^{-\frac{1}{k}+\varepsilon} \right). \quad (11)$$

Combining (9), (10) and (11) we have

$$\sum_{n \leq x} d(a(n)) = \frac{1}{kk!} \left( \frac{6}{k\pi^2} \right)^{k-1} A_0 x \ln^k x + A_1 x \ln^{k-1} x + \dots + A_{k-1} x \ln x + A_k x + O \left( x^{1-\frac{1}{2k}+\varepsilon} \right),$$

where  $A_0$  equals to  $B_0$ .

This proves Theorem 1.

Using the methods of proving Theorem 1 we can also prove Theorem 2. This completes the proof of the Theorems.

### Acknowledgments

The author expresses his gratitude to professor Zhang Wenpeng for his very helps and detailed instructions.

### REFERENCES:

1. F.Smarandache, Only problems, not Solutions, Xiquan Publ. House, Chicago, 1993, PP. 35.
2. T.M.Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.
3. Pan Chengdong and Pan Chengbiao, Foundation of Analytic Number Theory, Science Press, Beijing, 1997, PP. 98.
4. Zhang Wenpeng, On the cube part sequence of a positive integer (to appear).