On The Irrationality Of Certain Alternative Smarandache Series

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1. Let $S(n)$ be the Smarandache function. In paper [1] it is proved the irrationality of $\sum_{n=1}^{\infty}\frac{S(n)}{n!}$. We note here that this result is contained in the following more general theorem (see e.g. [2]).

Theorem 1 Let (x_n) be a sequence of natural numbers with the properties: (1) there exists $n_0 \in N^*$ such that $x_n \leq n$ for all $n \geq n_0$; (2) $x_n < n-1$ for an infinity of n; (3) $x_m > 0$ for infinitely many m. Then the series $\sum_{n=1}^{\infty}\frac{x_n}{n!}$ is irrational.

By letting $x_n = S(n)$, it is well known that $S(n) \leq n$ for $n \geq n_0 \equiv 1$, and $S(n) \leq \frac{2}{3}n$ for $n > 4$, composite. Clearly, $\frac{2}{3}n < n-1$ for $n > 3$. Thus the irrationality of the second constant of Smarandache ([1]) is contained in the above result.

2. We now prove a result on the irrationality of the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

We can formulate our result more generally, as follows:

Theorem 2 Let $(a_n),(b_n)$ be two sequences of positive integers having the following properties: (1) $n|a_1a_2\ldots a_n$ for all $n \geq n_0$ ($n_0 \in N^*$); (2) $\frac{b_{n+1}}{a_{n+1}} < b_n \leq a_n$ for $n \geq n_0$; (3) $b_m < a_m$, where $m \geq n_0$ is composite. Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{b_n}{a_1a_2\ldots a_n}$ is convergent and has an irrational value.

Proof: It is sufficient to consider the series $\sum_{n=n_0}^{\infty}(-1)^{n-1}\frac{b_n}{a_1a_2\ldots a_n}$. The proof is very similar (in some aspect) to Theorem 2 in our paper [3]. Let $x_n = \frac{b_n}{a_1a_2\ldots a_n}$ $(n \geq n_0)$.

Then $x_n \leq \frac{1}{a_1 \dots a_{n-1}} \to 0$ since (1) gives $a_1 \dots a_k \geq k \to \infty$ (as $k \to \infty$). On the other hand, $x_{n+1} < x_n$ by the first part of (2). Thus Leibnitz criteria assures the convergence of the series. Let us now assume, on the contrary, that the series has a rational value, say $\frac{a}{k}$. First we note that we can choose k in such a manner that $k+1$ is composite, and $k > n_0$. Indeed, if $k+1 = 1$ (prime), then $\frac{a}{p-1} = \frac{ca}{c(p-1)}$. Let $c = 2ar^2 + 2r$, where r is arbitrary. Then $2a(2ar^2 + 2r) + 1 = (2ar+1)^2$, which is composite. Since r is arbitrary, we can assume $k > n_0$. By multiplying the sum with $a_1 a_2 \dots a_k$, we can write:

$$a \frac{a_1 \dots a_k}{k} = \sum_{n=n_0}^{k} (-1)^{n-1} \frac{a_1 \dots a_k}{a_1 \dots a_n} \cdot b_n + (-1)^k \left(\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}} + \dots \right).$$

The alternating series on the right side is convergent and must have an integer value. But it is well known its value lies between $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}}$ and $\frac{b_{k+1}}{a_{k+1}}$. Here $\frac{b_{k+1}}{a_{k+1}} - \frac{b_{k+2}}{a_{k+1} a_{k+2}} > 0$ on base of (3). On the other hand $\frac{b_{k+1}}{a_{k+1}} < 1$, since $k+1$ is a composite number. Since an integer number has a value between 0 and 1, we have obtained a contradiction, finishing the proof of the theorem.

__Corollary__ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{S(n)}{n!}$ is irrational.

__Proof__: Let $a_n = n$. Then condition (1) of Theorem 2 is obvious for all n; (2) is valid with $n_0 = 2$, since $S(n) \leq n$ and $S(n+1) \leq n+1 = (n+1) \cdot 1 < (n+1) S(n)$ for $n \geq 2$.
For composite m we have $S(m) \leq \frac{2}{3} m < m$, thus condition (3) is verified, too.

References:

1. I. Cojocaru and S. Cojocaru ***The Second Constant Of Smarandache***, Smarandache Notions Journal, vol. 7, no. 1-2-3 (1996), 119-120

2. J. Sàndor ***Irrational Numbers***, Caiete metodico-ştiinţifice, no. 44, Universitatea din Timişoara, 1987, p. 1-18 (see p. 5)

3. J. Sàndor ***On The Irrationality Of Some Alternating Series***. Studia Univ Babeş-Bolay, Mathematica, XXXIII, 4, 1988, p. 7-12