# ON THE $k$-FULL NUMBER SEQUENCES

XU ZHEFENG

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

ABSTRACT. The main purpose of this paper is to study the asymptotic property of the $k$-full numbers (where $k \geq 2$ is a fixed integer), and obtain some interesting asymptotic formulas.

## 1. INTRODUCTION AND RESULTS

Let $k \geq 2$ is a fixed integer, a natural number $n$ is called a $k$-power free number if $p^k \nmid n$ for any prime $p$. If $p \mid n$ implies $p^k \mid n$, we call $n$ as a $k$-full number. In problem 31 of reference [1], Professor F. Smarandache asked us to study the properties of the $k$-power free number sequences. It is clear that there are some close relations between $k$-power free number sequences and $k$-full number sequences. In this paper, we use the analytic method to study the asymptotic properties of $k$-full number sequences, and obtain some interesting asymptotic formulas. That is, we shall prove the following six Theorems.

**Theorem 1.** *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} n = \frac{6k \cdot x^{1+\frac{1}{k}}}{(k+1)\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right) + O\left(x^{1+\frac{1}{2k}+\varepsilon}\right),$$

*where $\varepsilon$ denotes any fixed positive number.*

**Theorem 2.** *Let $\varphi(n)$ is the Euler function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} \varphi(n) = \frac{6k \cdot x^{1+\frac{1}{k}}}{(k+1)\pi^2} \prod_p \left(1 + \frac{p - p^{\frac{1}{k}}}{p^{2+\frac{1}{k}} - p^2 + p^{1+\frac{1}{k}} - p}\right) + O\left(x^{1+\frac{1}{2k}+\varepsilon}\right).$$

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**Theorem 3.** *Let $\alpha > 0$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \sigma_\alpha(n) = \frac{6k \cdot x^{\alpha+\frac{1}{k}}}{(k\alpha+1)\pi^2} \prod_p \left(1 + \frac{p^{\alpha+\frac{1}{k}}(p^{\frac{1}{k}}-1)\sum_{i=1}^{k}\left(\frac{1}{p^i}\right)^\alpha + p^{\alpha+\frac{1}{k}} + p^{\frac{1}{k}} - 1}{\left(p^{\alpha+\frac{1}{k}}-1\right)(p+1)(p^{\frac{1}{k}}-1)}\right)$$
$$+ O\left(x^{\alpha+\frac{1}{2k}+\varepsilon}\right).$$

**Theorem 4.** *Let $d(n)$ denotes Dirichlet divisor function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} d(n) = \frac{6k \cdot x^{\frac{1}{k}}}{\pi^2} \prod_p \left(1 + \frac{(2p^{\frac{1}{k}}-1)\sum_{i=2}^{k+1}\binom{k+1}{i}p^{k+1-i} - kp^{k+\frac{1}{k}}}{(p+1)^{k+1}(p^{\frac{1}{k}}-1)^2}\right) \cdot f(\log x)$$
$$+ O\left(x^{\frac{1}{2k}+\varepsilon}\right).$$

*where $f(y)$ is a polynomial of $y$ with degree $k$.*

**Theorem 5.** *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \sigma_\alpha((m,n)) = \frac{6k \cdot x^{\frac{1}{k}}}{\pi^2} \prod_{p\nmid m}\left(1 + \frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right) \prod_{\substack{p^\beta \| m\\ \beta\leq k}}\left(1 + \frac{p^{\frac{1}{k}}\sum_{i=0}^{\beta}p^{i\alpha}}{p(p^{\frac{1}{k}}-1)}\right)$$
$$\times \prod_{\substack{p^\beta \| m\\ \beta > k}}\left(1 + \sum_{i=k}^{\beta-1}p^{-\frac{i}{k}}\sum_{j=0}^{i}p^{j\alpha} + \frac{p^{\frac{1}{k}}\sum_{i=0}^{\beta}p^{i\alpha}}{p(p^{\frac{1}{k}}-1)}\right)\prod_{p|m}\left(\frac{p}{p+1}\right) + O\left(x^{\frac{1}{2k}+\varepsilon}\right).$$

*where $m$ is any fixed integer, $(m,n)$ denotes greatest common divisor of $m$ and $n$.*

**Theorem 6.** *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \sigma_\alpha((m,n)) = \frac{6k \cdot x^{\frac{1}{k}}}{\pi^2} \prod_{p\nmid m}\left(1 + \frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right) \prod_{\substack{p^\beta \| m\\ \beta\leq k}}\left(1 + \frac{\left(p^\beta - p^{\beta-1}\right)p^{\frac{1}{k}}}{p(p^{\frac{1}{k}}-1)}\right)$$
$$\times \prod_{\substack{p^\beta \| m\\ \beta > k}}\left(1 + \sum_{i=k}^{\beta-1}p^{-\frac{i}{k}}\left(p^i - p^{i-1}\right) + \frac{\left(p^\beta - p^{\beta-1}\right)p^{\frac{1}{k}}}{p(p^{\frac{1}{k}}-1)}\right)\prod_{p|m}\left(\frac{p}{p+1}\right) + O\left(x^{\frac{1}{2k}+\varepsilon}\right).$$

## 2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. For conveniently we define a new number theory function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } n = 1; \\ n, & \text{if } n \text{ is a } k\text{-full number} \\ 0, & \text{if } n \text{ is not a } k\text{-full number} \end{cases}$$

It is clear that

$$\sum_{\substack{n\in A\\ n\le x}} n = \sum_{n\le x} a(n).$$

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $a(n)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{a(p^k)}{p^{ks}} + \frac{a(p^{k+1})}{p^{(k+1)s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^{k(s-1)}} \frac{1}{1-\frac{1}{p^{s-1}}}\right) \\ &= \prod_p \left(1 + \frac{1}{p^{k(s-1)}}\right) \prod_p \left(1 + \frac{1}{(p^{k(s-1)}+1)(p^{s-1}-1)}\right) \\ &= \frac{\zeta(k(s-1))}{\zeta(2k(s-1))} \prod_p \left(1 + \frac{1}{(p^{k(s-1)}+1)(p^{s-1}-1)}\right), \end{aligned}$$

where $\zeta(s)$ is Riemann zeta function. Obviously, we have inequality

$$|a(n)| \le n, \qquad \left|\sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma}\right| < \frac{1}{\sigma - 1 - \frac{1}{k}},$$

where $\sigma > 1 - \frac{1}{k}$ is the real part of $s$. So by Perron formula [3]

$$\begin{aligned} \sum_{n\le x} \frac{a(n)}{n^{s_0}} &= \frac{1}{2i\pi} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\ &\quad + O\left(x^{1-\sigma_0} H(2x) \min(1, \frac{\log x}{T})\right) + O\left(x^{-\sigma_0} H(N) \min(1, \frac{x}{||x||})\right), \end{aligned}$$

where $N$ is the nearest integer to $x$, $||x|| = |x - N|$. Taking $s_0 = 0$, $b = 2 + \frac{1}{k}$, $T = x^{1+\frac{1}{2k}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma - 1 - \frac{1}{k}}$, we have

$$\sum_{n\le x} a(n) = \frac{1}{2i\pi} \int_{2+\frac{1}{k}-iT}^{2+\frac{1}{k}+iT} \frac{\zeta(k(s-1))}{\zeta(2k(s-1))} R(s) \frac{x^s}{s} ds + O(x^{1+\frac{1}{2k}+\epsilon}),$$

where

$$R(s) = \prod_p \left(1 + \frac{1}{(p^{k(s-1)}+1)(p^{s-1}-1)}\right).$$

To estimate the main term

$$\frac{1}{2i\pi} \int_{2+\frac{1}{k}-iT}^{2+\frac{1}{k}+iT} \frac{\zeta(k(s-1))x^s}{\zeta(2k(s-1))s} R(s) ds,$$

we move the integral line from $s = 2 + \frac{1}{k} \pm iT$ to $s = 1 + \frac{1}{2k} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(k(s-1))x^s}{\zeta(2k(s-1))s}R(s)$$

have a simple pole point at $s = 1 + \frac{1}{k}$ with residue $\frac{kx^{1+\frac{1}{k}}}{(k+1)\zeta(2)}R(1+\frac{1}{k})$. So we have

$$\frac{1}{2i\pi}\left(\int_{2+\frac{1}{k}-iT}^{2+\frac{1}{k}+iT} + \int_{2+\frac{1}{k}+iT}^{1+\frac{1}{2k}+iT} + \int_{1+\frac{1}{2k}+iT}^{1+\frac{1}{2k}-iT} + \int_{1+\frac{1}{2k}-iT}^{2+\frac{1}{k}-iT}\right)\frac{\zeta(k(s-1))x^s}{\zeta(2k(s-1))s}R(s)ds$$
$$= \frac{k \cdot x^{1+\frac{1}{k}}}{(k+1)\zeta(2)}\prod_p\left(1 + \frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right).$$

We can easy get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{2+\frac{1}{k}+iT}^{1+\frac{1}{2k}+iT} + \int_{1+\frac{1}{2k}-iT}^{2+\frac{1}{2}-iT}\right)\frac{\zeta(k(s-1))x^s}{\zeta(2k(s-1))s}R(s)ds\right|$$
$$\ll \int_{1+\frac{1}{2k}}^{2+\frac{1}{k}}\left|\frac{\zeta(k(\sigma-1+iT))}{\zeta(2k(\sigma-1+iT))}R(s)\frac{x^{2+\frac{1}{k}}}{T}\right|d\sigma \ll \frac{x^{2+\frac{1}{k}}}{T} = x^{1+\frac{1}{2k}}$$

and

$$\left|\frac{1}{2\pi i}\int_{1+\frac{1}{2k}+iT}^{1+\frac{1}{2k}-iT}\frac{\zeta(k(s-1))x^s}{\zeta(2k(s-2))s}R(s)ds\right| \ll \int_0^T\left|\frac{\zeta(1/2+ikt)}{\zeta(1+2ikt)}\frac{x^{1+\frac{1}{2k}}}{t}\right|dt \ll x^{1+\frac{1}{2k}+\varepsilon}.$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the above we have

$$\sum_{\substack{n\in A\\ n\le x}} n = \frac{6k \cdot x^{1+\frac{1}{k}}}{(k+1)\pi^2}\prod_p\left(1+\frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right) + O\left(x^{1+\frac{1}{2k}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Let

$$f_1(s) = \sum_{\substack{n=1\\ n\in A}}^{\infty}\frac{\varphi(n)}{n^s}, \qquad f_2(s) = \sum_{\substack{n=1\\ n\in A}}^{\infty}\frac{\sigma_\alpha(n)}{n^s}, \qquad f_3(s) = \sum_{\substack{n=1\\ n\in A}}^{\infty}\frac{d(n)}{n^s},$$

$$f_4(s) = \sum_{\substack{n=1\\ n\in A}}^{\infty}\frac{\sigma_\alpha((m,n))}{n^s}, \qquad f_5(s) = \sum_{\substack{n=1\\ n\in A}}^{\infty}\frac{\varphi((m,n))}{n^s}.$$

From the Euler product formula [2] and the definition of $\varphi(n), \sigma_\alpha(n)$ and $d(n)$, we also have

$$f_1(s) = \prod_p\left(1+\frac{\varphi(p^k)}{p^{ks}}+\frac{\varphi(p^{k+1})}{p^{(k+1)s}}+\cdots\right) = \prod_p\left(1+\frac{\varphi(p^k)}{p^{ks}}\left(\frac{1}{1-\frac{1}{p^{s-1}}}\right)\right)$$
$$= \frac{\zeta(k(s-1))}{\zeta(2k(s-1))}\prod_p\left(1+\frac{p-p^{s-1}}{(p^{k(s-1)}+1)(p^s-p)}\right);$$

$$f_2(s) = \frac{\zeta(k(s-\alpha))}{\zeta(2k(s-\alpha))} \prod_p \left(1 + \frac{(p^{s-\alpha}-1)p^s \sum_{i=1}^{k} \left(\frac{1}{p^i}\right)^\alpha + p^s + p^{s-\alpha} - 1}{(p^{k(s-\alpha)}+1)(p^{s-\alpha}-1)(p^s-1)}\right);$$

$$f_3(s) = \frac{\zeta^{k+1}(ks)}{\zeta^{k+1}(2ks)} \prod_p \left(1 + \frac{(2p^s-1)\sum_{i=2}^{k+1}\binom{k+1}{i} p^{k(k+1-i)s} - kp^{(k^2+1)s}}{(p^{ks}+1)^{k+1}(p^s-1)^2}\right);$$

$$\begin{aligned} f_4(s) &= \prod_p \left(1 + \frac{\sigma_\alpha((m,p^k))}{p^{ks}} + \frac{\sigma_\alpha((m,p^{k+1}))}{p^{(k+1)s}} + \cdots\right) \\ &= \frac{\zeta(ks)}{\zeta(2ks)} \prod_{p|m} \left(\frac{p^{ks}}{p^{ks}+1}\right) \prod_{p\nmid m} \left(1 + \frac{1}{(p^{ks}+1)(p^s-1)}\right) \\ &\quad \times \prod_{\substack{p^\beta \| m \\ \beta \le k}} \left(1 + \frac{\sigma_\alpha(p^\beta)}{p^{ks}(1-\frac{1}{p^s})}\right) \prod_{\substack{p^\beta \| m \\ \beta > k}} \left(1 + \sum_{i=k}^{\beta-1} \frac{\sigma_\alpha(p^i)}{p^{is}} + \frac{\sigma_\alpha(p^\beta)}{p^{ks}(1-\frac{1}{p^s})}\right) \end{aligned}$$

and

$$\begin{aligned} f_5(s) &= \frac{\zeta(ks)}{\zeta(2ks)} \prod_{p|m} \left(\frac{p^{ks}}{p^{ks}+1}\right) \prod_{p\nmid m} \left(1 + \frac{1}{(p^{ks}+1)(p^s-1)}\right) \\ &\quad \times \prod_{\substack{p^\beta \| m \\ \beta \le k}} \left(1 + \frac{p^\beta - p^{\beta-1}}{p^{ks}(1-\frac{1}{p^s})}\right) \prod_{\substack{p^\beta \| m \\ \beta > k}} \left(1 + \sum_{i=k}^{\beta-1} \frac{p^i - p^{i-1}}{p^{is}} + \frac{p^\beta - p^{\beta-1}}{p^{ks}(1-\frac{1}{p^s})}\right). \end{aligned}$$

By Perron formula [3] and the method of proving Theorem 1, we can obtain the other results.

## References


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