ON THE MEAN VALUE OF m-TH POWER FREE NUMBERS*

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ABSTRACT. A positive integer n is called m-th power free number if it can not be divided by m-th power of each prime. The main purpose of this paper is to give a k-th power mean value formula for the m-th power free numbers.

1. INTRODUCTION

A positive integer n is called m-th power free number if it is not multiples of 2^m, 3^m, 5^m, $7^m \cdots p^m$ and so on. That is, it can not be divided by m-th power of each prime. Generally, one obtains all m-th power free numbers if he takes off all multiples of m-power primes from the set of natural numbers (except 0 and 1). Let $a(n,m)$ denotes the m-power free sequence . In problem 31 of [1], Professor F.Smarandach asked us to study the properties of this sequence. In this paper, we use the elementary methods to study the mean value properties of this sequence, and give its k-th power mean value formula. That is, we shall prove the following main conclusion:

Theorem. *For any positive integer $x > 1$ and $n > 0$, we have the asymptotic formula*

$$\sum_{n\leq x} a^k(n,m) = \begin{cases} \frac{x^{k+1}}{k+1}\frac{1}{\zeta(m)} + O(x^{k+\frac{1}{m}}) & \text{if } k \geq 0; \\ \frac{x^{k+1}}{k+1}\frac{1}{\zeta(m)} + \frac{\zeta(-k)}{\zeta(-mk)} + O(x^{k+\frac{1}{m}}) & \text{if } k < 0 \text{ but } k \neq -1; \\ \frac{\log x}{\zeta(m)} - m\frac{\zeta'(m)}{\zeta^2(m)} + O(x^{1-m}) & \text{if } k = -1. \end{cases}$$

where $\zeta(m)$ is the Riemman-zeta function.

If $m = 3$, then the sequence become the cube free number, from our theorem we may deduce the following:

Corollary. *For any positive integer $x > 1$ and $n > 0$, we have the formula*

$$\sum_{n\leq x} a^k(n,3) = \begin{cases} \frac{x^{k+1}}{k+1}\frac{1}{\zeta(3)} + O(x^{k+\frac{1}{3}}) & \text{if } k \geq 0; \\ \frac{x^{k+1}}{k+1}\frac{1}{\zeta(3)} + \frac{\zeta(-k)}{\zeta(-3k)} + O(x^{k+\frac{1}{3}}) & \text{if } k < 0 \text{ but } k \neq -1; \\ \frac{\log x}{\zeta(3)} - 3\frac{\zeta'(3)}{\zeta^2(3)} + O(x^{-2}) & \text{if } k = -1. \end{cases}$$

Key words and phrases. M-th power free numbers; Mean value property; Asymptotic formula.
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2. Proof of the theorem

In this section, we shall use elementary methods and the Euler summation formula to complete the proof of the theorem. First for any positive integer $x > 1$ and $n > 0$, if $k \geq 0$, we have the asymptotic formula

$$\begin{aligned}
\sum_{n \leq x} a^k(n,m) &= \sum_{n \leq x} n^k \sum_{d^m | n} \mu(d) = \sum_{d \leq x^{1/m}} d^{mk} \mu(d) \sum_{n \leq x/d^m} n^k \\
&= \sum_{d \leq x^{1/m}} d^{mk} \mu(d) \left(\frac{\left(\frac{x}{d^m}\right)^{k+1}}{k+1} + O\left(\frac{x^k}{d^{mk}}\right) \right) \\
&= \frac{x^{k+1}}{k+1} \sum_{d \leq x^{1/m}} \frac{\mu(d)}{d^m} + O(x^{k+1/m}) \\
&= \frac{x^{k+1}}{k+1} \frac{1}{\zeta(m)} + O(x^{k+1/m})
\end{aligned}$$

Furthermore, if $k < 0$ but $k \neq -1$, we have

$$\begin{aligned}
\sum_{n \leq x} a^k(n,m) &= \sum_{d \leq x^{1/m}} d^{mk} \mu(d) \sum_{n \leq x/d^m} \frac{1}{n^{-k}} \\
&= \sum_{d \leq x^{1/m}} d^{mk} \mu(d) \left(\frac{\left(\frac{x}{d^m}\right)^{k+1}}{k+1} + \zeta(-k) + O\left(\frac{x^k}{d^{mk}}\right) \right) \\
&= \sum_{d \leq x^{1/m}} \frac{x^{1+k}}{1+k} \frac{\mu(d)}{d^m} + \sum_{d \leq x^{1/m}} \zeta(-k) d^{mk} \mu(d) + O(x^{k+1/m}) \\
&= \frac{x^{1+k}}{1+k} \frac{1}{\zeta(m)} + \frac{\zeta(-k)}{\zeta(-mk)} + O(x^{k+1/m})
\end{aligned}$$

Note that

$$F(m) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^m} = \frac{1}{\zeta(m)}$$

we have

$$F'(m) = -\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^m} = -\frac{\zeta'(m)}{\zeta^2(m)}$$

From this formula, we can immediately get

$$\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^m} = \frac{\zeta'(m)}{\zeta^2(m)}$$

If $k = -1$, we have

$$\begin{aligned}
\sum_{n\le x}\frac{1}{a(n,m)} &= \sum_{d\le x^{1/m}} d^{-m}\mu(d)\sum_{n\le x/d^m}\frac{1}{n}\\
&= \sum_{d\le x^{1/m}}\frac{\mu(d)}{d^m}\log\frac{x}{d^m}\\
&= \sum_{d\le x^{1/m}}\frac{\mu(d)}{d^m}(\log x - m\log d)\\
&= \frac{\log x}{\zeta(m)} - m\sum_{n=1}^{\infty}\frac{\mu(d)\log d}{d^m} + O(x^{1-m})\\
&= \frac{\log x}{\zeta(m)} - m\frac{\zeta'(m)}{\zeta^2(m)} + O(x^{1-m})
\end{aligned}$$

This completes the proof of the Theorem.

References

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