ON THE MEAN VALUE OF SMARANDACHE DOUBLE FACTORIAL FUNCTION*

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ABSTRACT. For any positive integer n, the Smarandache double factorial function $d_f(n)$ is defined to be the smallest integer such that $d_f(n)!!$ is a multiple of n. In this paper, we study the hybrid mean value of the Smarandache double factorial function and the Mangoldt function, and give a sharp asymptotic formula.

1. INTRODUCTION AND RESULTS

For any positive integer n, the Smarandache double factorial function $d_f(n)$ is defined to be the smallest integer such that $d_f(n)!!$ is a factorial number. For example, $d_f(1) = 1$, $d_f(2) = 2$, $d_f(3) = 3$, $d_f(4) = 4$, $d_f(5) = 5$, $d_f(6) = 6$, $d_f(7) = 7$, $d_f(8) = 4$, $\cdots$. Professor F. Smarandache [1] asks us to study the sequence. About this problem, we know very little. There are many papers on the Smarandache double factorial function. For example, some arithmetic properties of this sequence are studied by C.Dumitrescu, V. Seleacu [2] and Felice Russo [3], [4]. The problem is interesting because it can help us to calculate the Smarandache function.

In this paper, we study the hybrid mean value of the Smarandache double factorial function and the Mangoldt function, and give a sharp asymptotic formula. That is, we shall prove the following theorems.

Theorem 1. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n<x} \Lambda_1(n) d_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \text{if } n \text{ is a prime } p\ ; \\ 0, & \text{otherwise,} \end{cases}$$

and $a_m (m = 1, 2, \cdots, k-1)$ are computable constants.

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Theorem 2. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n\leq x} \Lambda(n) d_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where $\Lambda(n)$ is the Mangoldt function.

2. Some Lemmas

To complete the proofs of the theorems, we need the following lemma.

Lemma 1. *For any positive integer α, if $p \geq (2\alpha - 1)$ we have*

$$d_f(p^\alpha) = (2\alpha - 1)p.$$

Proof. This is Theorem 5 of [4].

3. Proofs of the Theorems

In this section, we complete the proofs of the theorems. Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise,} \end{cases}$$

then for any positive integer k we have

$$\sum_{n\leq x} a(n) = \pi(x) = \frac{x}{\log x} \left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x} \right) + O\left(\frac{x}{\log^{k+1} x} \right).$$

By Abel's identity we have

$$\begin{aligned}
\sum_{n\leq x} \Lambda_1 d_f(n) &= \sum_{p\leq x} p \log p = \sum_{n\leq x} a(n) n \log n = \pi(x) \cdot x \log x - \int_2^x \pi(t) (\log t + 1)\, dt \\
&= x^2 \left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x} \right) + O(\frac{x^2}{\log^k x}) \\
&\quad - \int_2^x \left(t + \frac{t}{\log t} + t \sum_{m=1}^{k-1} \frac{m!}{\log^m t} + \frac{t}{\log t} \sum_{m=1}^{k-1} \frac{m!}{\log^m t} + O\left(\frac{t\,(\log t + 1)}{\log^{k+1} t} \right) \right) dt \\
&= x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),
\end{aligned}$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants. Therefore

$$\sum_{p\leq x} p \log p = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

So we have

$$\sum_{n<x} \Lambda_1(n) d_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

This proves Theorem 1.

It is obvious that $d_f(p^\alpha) \leq (2\alpha - 1)p$. From Lemma 1 we have

$$\sum_{n<x} \Lambda(n) d_f(n) = \sum_{p^\alpha < x} \log p \left[(2\alpha - 1)p \right] + \sum_{\substack{p^\alpha < x \\ p < (2\alpha - 1)}} \log p \left[d_f(p^\alpha) - (2\alpha - 1)p \right].$$

Note that

$$\sum_{p^\alpha \leq x} (2\alpha - 1) p \log p - \sum_{p^\alpha \leq x} p \log p = \sum_{\alpha \leq \frac{\log x}{\log p}} \sum_{p \leq x^{1/\alpha}} p \log p (2\alpha - 1) - \sum_{p^\alpha \leq x} p \log p$$

$$= \sum_{2 \leq \alpha \leq \frac{\log x}{\log p}} \sum_{p \leq x^{1/\alpha}} p \log p (2\alpha - 1) \ll \sum_{2 \leq \alpha \leq \frac{\log x}{\log p}} \alpha x^{2/\alpha} \log x^{1/\alpha} \ll x \log^3 x$$

and

$$\sum_{\substack{p^\alpha < x \\ p < (2\alpha - 1)}} \log p \left[d_f(p^\alpha) - (2\alpha - 1)p \right] \ll \sum_{\alpha < \frac{\log x}{\log 2}} \sum_{p < (2\alpha - 1)} \alpha p \log p$$

$$\ll \sum_{\alpha < \frac{\log x}{\log 2}} (2\alpha - 1)^2 \alpha \log(2\alpha - 1) \ll \log^3 x,$$

so we have

$$\sum_{n \leq x} \Lambda(n) d_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right).$$

This completes the proof of Theorem 2.

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