On the numerical function $S_{\min}^{-1}$

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In [1] on defines $S_{\min}^{-1} : \mathrm{N} \setminus \{1\} \mapsto \mathrm{N}$, $S_{\min}^{-1}(x) = \min\{S^{-1}(x)\}$, where $S^{-1}(x) = \{a \in \mathrm{N} \mid S(a) = x\}$, and S is the Smarandache function.
For example $S^{-1}(6) = \{2^4,\ 2^4 \cdot 3,\ 2^4 \cdot 3^2,\ 3^2,\ 3^2 \cdot 2,\ 3^2 \cdot 2^2,\ 3^2 \cdot 2^3,\ 2^4 \cdot 3 \cdot 5,\ 2^3 \cdot 3^2 \cdot 5,\ 2^4 \cdot 3^2 \cdot 5,\ 3^2 \cdot 5,\ 2^4 \cdot 5,\ 3^2 \cdot 5,\ 3^2 \cdot 2^4\}$ and $S_{\min}^{-1}(6) = 3^2$.
If $S(x) = n$ one knows that $card\,(S^{-1}(n)) = d(n!) - d((n-1)!)$ where d is the number of divisors of n.
If x is a prime number, then $card\,(S^{-1}(n)) = d((n-1)!)$.
We give below a table of the values of $S_{\min}^{-1}(n)$:

n	2	3	4	5	6	7	8	12	15
$S_{\min}^{-1}(n)$	2	3	4	5	3^2	7	2^5	3^5	5^3
n	16	21	24	27	36	40	52	56	60
$S_{\min}^{-1}(n)$	2^{12}	7^3	3^{10}	3^{11}	3^{16}	5^9	13^4	7^8	5^4

One knows [2] that if $p < q$ are two prime numbers, and $n > 1$ is a natural number such that $p \cdot q \mid n$, then $p^{l_p(n)} > q^{l_q(n)}$, where $l_p(n)$ is the exponent of p in the prime factors decomposition of $n!$.
According to the above properties we can deduce the calculus formula for function $S_{\min}^{-1}$:

$$S_{\min}^{-1}(n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_r^{l_{p_r}(n) - \alpha_r + 1} \tag{1}$$

where $p_1 < p_2 < \cdots < p_r$ are the prime numbers in the canonical decomposition of the number n.
We list a set of properties of the function $S_{\min}^{-1}$, which result directly from the definition and from formula (1):

1. $S_{\min}^{-1}(p) = p$ if p is a prime number.

2. $S_{\min}^{-1}(p \cdot q) = q^p$ if p and q are prime numbers and $p < q$.

3. $S\left(S_{\min}^{-1}(x)\right) = x$.

4. $S_{\min}^{-1}(q^p) = p \cdot q$ if p and q are prime numbers and $p < q$.

5. $S_{\min}^{-1}(x) < S_{\min}^{-1}(y)$ if x and y contain as the greatest prime factor p_r and $x < y$.

6. The equation $S_{\min}^{-1}(x) = S_{\min}^{-1}(x+1)$ has not solutions.

7. $S_{\min}^{-1}(S(x))$ is generally not equal to $S(x)$.

8. $\Lambda\left(S_{\min}^{-1}(x)\right) = \log p_r$, where Λ is the Mangoltd function.

It is open the problem to find other properties of the function $S_{\min}^{-1}$.

References

[1] Smarandache Function Journal, vol. 1, 1993.

[2] Florian Luca, *An inequality between prime powers dividing n!* (To appear in Smarandache Function Journal, vol. 9, 1998.)

[3] Ion Bălăcenoiu, *Remarkable Inequalities*, Proceedings of the first International Conference on Smarandache Type Notions in Number Theory, Craiova, Romania, 1997.