ON THE PERMUTATION SEQUENCE AND ITS SOME PROPERTIES*

ZHANG WENPENG

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

ABSTRACT. The main purpose of this paper is to prove that there is no any perfect power among the permutation sequence: 12, 1342, 135642, 13578642, 13579108642, $\cdots\cdots$. This answered the question 20 of F.Smarandach in [1].

for $n \leq 9$ (partially)

1. INTRODUCTION

For any positive integer n, we define the permutation sequence $\{P(n)\}$ as follows: $P(1) = 12$, $P(2) = 1342$, $P(3) = 135642$, $P(4) = 13578642$, $P(5) = 13579108642$, $\cdots\cdots$, $P(n) = 135\cdots(2n-1)(2n)(2n-2)\cdots 42$, $\cdots\cdots$,. In problem 20 of [1], Professor F.Smarandach asked us to answer such a question: Is there any perfect power among these numbers? Conjecture: no! This problem is interesting, because it can help us to find some new properties of permutation sequence. In this paper, we shall study the properties of the permutation sequence $P(n)$, and proved that the F.Smarandach conjecture is true. This solved the problem 20 of [1], and more, we also obtained some new divisible properties of $P(n)$. That is, we shall prove the following conclusion:

Theorem. *There is no any perfect power among permutation sequence, and*

$$P(n) = \frac{1}{81}\left(11 \cdot 10^{2n} - 13 \cdot 10^n + 2\right) = \overbrace{11\cdots1}^{n} \times 1\overbrace{22\cdots2}^{n}, \text{ for } n \leq 9.$$

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any positive integer n, we have

$$\begin{aligned} P(n) &= 10^{2n-1} + 3 \times 10^{2n-2} + \cdots + (2n-1) \times 10^n \\ &\quad + 2n \times 10^{n-1} + (2n-2) \times 10^{n-2} + \cdots 4 \times 10 + 2 \\ &= \left[10^{2n-1} + 3 \times 10^{2n-2} + \cdots + (2n-1) \times 10^n\right] \\ &\quad + \left[2n \times 10^{n-1} + (2n-2) \times 10^{n-2} + \cdots 4 \times 10 + 2\right] \end{aligned}$$

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$$(1) \qquad \equiv S_1 + S_2.$$

Now we compute S_1 and S_2 in (1) respectively. Note that

$$\begin{aligned}
9S_1 = 10S_1 - S_1 &= 10^{2n} + 3 \times 10^{2n-1} + \cdots (2n-1) \times 10^{n+1} \\
&\quad - 10^{2n-1} - 3 \times 10^{2n-2} - \cdots - (2n-1) \times 10^n \\
&= 10^{2n} + 2 \times 10^{2n-1} + 2 \times 10^{2n-2} + \cdots + 2 \times 10^{n+1} - (2n-1) \times 10^n \\
&= 10^{2n} + 2 \times 10^{n+1} \times \frac{10^{n-1}-1}{9} - (2n-1) \times 10^n
\end{aligned}$$

and

$$\begin{aligned}
9S_2 = 10S_2 - S_2 &= 2n \times 10^n + (2n-2) \times 10^{n-1} + \cdots 4 \times 10^2 + 2 \times 10 \\
&\quad - 2n \times 10^{n-1} - (2n-2) \times 10^{n-2} - \cdots 4 \times 10 - 2 \\
&= 2n \times 10^n - 2 \times 10^{n-1} - 2 \times 10^{n-2} - \cdots 2 \times 10 - 2 \\
&= 2n \times 10^n - 2 \times \frac{10^n - 1}{9}.
\end{aligned}$$

So that

$$(2) \qquad S_1 = \frac{1}{81} \times \left[11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n\right]$$

and

$$(3) \qquad S_2 = \frac{1}{81}\left[18n \times 10^n - 2 \times 10^n + 2\right].$$

Thus combining (1), (2) and (3) we have

$$\begin{aligned}
P(n) = S_1 + S_2 &= \frac{1}{81} \times \left[11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n\right] \\
&\quad + \frac{1}{81}\left[18n \times 10^n - 2 \times 10^n + 2\right]
\end{aligned}$$

$$(4) \qquad = \frac{1}{81}\left(11 \cdot 10^{2n} - 13 \cdot 10^n + 2\right) = \overbrace{11\cdots1}^{n} \times 1\overbrace{22\cdots2}^{n}.$$

From (4) we can easily find that $2 \mid P(n)$, but $4 \nmid P(n)$, if $n \geq 2$, (and $n \leq 9$.) So that $P(n)$ can not be a perfect power, if $n \geq 2$. In fact, if we assume $P(n)$ be a perfect power, then $P(n) = m^k$, for some positive integer $m \geq 2$ and $k \geq 2$. Since $2 \mid P(n)$, so that m must be an even number. Thus we have $4 \mid P(n)$. This contradiction with $4 \nmid P(n)$, if $n \geq 2$. Note that $P(1)$ is not a perfect power, so that $P(n)$ can be a perfect power for all $n \geq 1$, (and $n \leq 9$.) This completes the proof of the Theorem.

References

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