ON THE PRIMITIVE NUMBERS OF POWER P AND ITS ASYMPTOTIC PROPERTY*

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ABSTRACT. Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. In this paper, we study the asymptotic properties of $S_p(n)$, and give an interesting asymptotic formula for it.

1. INTRODUCTION

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F.Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to calculate the Smarandache function. In this paper, we use the elementary methods to study the asymptotic properties of $S_p(n)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any fixed prime p and any positive integer n, we have the asymptotic formula*

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

From this theorem we may immediately deduce the following:

Corollary. *For any positive integer n, we have the asymptotic formulas*

$$a) \quad S_2(n) = n + O(\ln n);$$

$$b) \quad S_3(n) = 2n + O(\ln n).$$

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2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First for any fixed prime p and any positive integer n, we let $a(n,p)$ denote the sum of the base p digits of n. That is, if $n = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$, then $a(n,p) = \sum_{i=1}^{s} a_i$, and for this number theoretic function, we have the following two simple Lemmas:

Lemma 1. *For any integer $n \geq 1$, we have the identity*

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}\left(n - a(n,p)\right),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From the properties of $[x]$ we know that

$$\begin{aligned}\left[\frac{n}{p^i}\right] &= \left[\frac{a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}}{p^i}\right] \\ &= \begin{cases} \sum_{j=k}^{s} a_jp^{\alpha_j - i}, & \text{if } \alpha_{k-1} < i \leq \alpha_k \\ 0, & \text{if } i > \alpha_s. \end{cases}\end{aligned}$$

So from this formula we have

$$\begin{aligned}\alpha(n) &\equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \sum_{i=1}^{+\infty} \left[\frac{a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}}{p^i}\right] \\ &= \sum_{j=1}^{s} \sum_{k=1}^{\alpha_j} a_jp^{\alpha_j - k} = \sum_{j=1}^{s} a_j\left(1 + p + p^2 + \cdots + p^{\alpha_j - 1}\right) \\ &= \sum_{j=1}^{s} a_j \cdot \frac{p^{\alpha_j} - 1}{p-1} = \frac{1}{p-1} \sum_{j=1}^{s} \left(a_jp^{\alpha_j} - a_j\right) \\ &= \frac{1}{p-1}\left(n - a(n,p)\right).\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n with $p|n$, we have the estimate*

$$a(n,p) \leq \frac{p}{\ln p} \ln n.$$

Proof. Let $n = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 1$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$. Then from the definition of $a(n,p)$ we have

$$a(n,p) = \sum_{i=1}^{s} a_i \leq \sum_{i=1}^{s} (p-1) = (p-1)s. \tag{1}$$

On the other hand, using the mathematical induction we can easily get the inequality

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s} \geq a_s p^s,$$

or

$$(2) \qquad s \leq \frac{\ln(n/a_s)}{\ln p} \leq \frac{\ln n}{\ln p}.$$

Combining (1) and (2) we immediately get the estimate

$$a(n,p) \leq \frac{p}{\ln p} \ln n.$$

This proves the Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any fixed prime p and any positive integer n, let $S_p(n) = k = a_1 \cdot p^{\alpha_1} + a_2 \cdot p^{\alpha_2} + \cdots + a_s \cdot p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$ under the base p. Then from the definition of $S_p(n)$ we know that $p^n | k!$ and $p^n \nmid (k-1)!$, so that $\alpha_1 \geq 1$. Note that the factorization of $k!$ into prime powers is

$$k! = \prod_{q \leq k} q^{\alpha_q(k)},$$

where $\prod_{q \leq k}$ denotes the product over all prime $\leq k$, and $\alpha_q(k) = \sum_{i=1}^{+\infty} \left[\frac{k}{q^i}\right]$. From Lemma 1 we immediately get the inequality

$$\alpha_p(k) - \alpha_1 < n \leq \alpha_p(k)$$

or

$$\frac{1}{p-1}(k - a(k,p)) - \alpha_1 < n \leq \frac{1}{p-1}(k - a(k,p)).$$

i.e.

$$(p-1)n + a(k,p) \leq k \leq (p-1)n + a(k,p) + (p-1)(\alpha_1 - 1).$$

Combining this inequality and Lemma 2 we obtain the asymptotic formula

$$k = (p-1)n + O\left(\frac{p}{\ln p} \ln k\right) = (p-1)n + O\left(\frac{p}{\ln p} \ln n\right).$$

This completes the proof of the Theorem.

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