On the Pseudo-Smarandache Function

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Kashihara[2] defined the Pseudo-Smarandache function Z by

$$Z(n) = \min\left\{ m \geq 1 : n \mid \frac{m(m+1)}{2} \right\}$$

Properties of this function have been studied in [1], [2] etc.

1. By answering a question by C. Ashbacher, Maohua Le proved that $S(Z(n)) - Z(S(n))$ changes signs infinitely often. Put

$$\Delta_{s,z}(n) = | S(Z(n)) - Z(S(s)) |$$

We will prove first that

$$\liminf_{n \to \infty} \Delta_{s,z}(n) \leq 1 \qquad (1)$$

and

$$\limsup_{n \to \infty} \Delta_{s,z}(n) = +\infty \qquad (2)$$

Indeed, let $n = \frac{p(p+1)}{2}$, where p is an odd prime. Then it is not difficult to see that

$S(n) = p$ and $Z(n) = p$. Therefore,

$$| S(Z(n)) - Z(S(n)) | = | S(p) - S(p) | = | p - (p-1) | = 1$$

implying (1). We note that if the equation $S(Z(n)) = Z(S(n))$ has infinitely many solutions, then clearly the lim inf in (1) is 0, otherwise is 1, since

$$| S(Z(n)) - Z(S(n)) | \geq 1,$$

$S(Z(n)) - Z(S(n))$ being an integer.

Now let $n = p$ be an odd prime. Then, since $Z(p) = p-1$, $S(p) = p$ and $S(p-1) \leq \frac{p-1}{2}$

(see [4]) we get

$$\Delta_{s,z}(p) = \left| S(p-1) - (p-1) \right| = p-1 - S(p-1) \geq \frac{p-1}{2} \to \infty \text{ as } p \to \infty$$

proving (2). Functions of type $\Delta_{f,g}$ have been studied recently by the author [5] (see also [3]).

2. Since $n \mid \frac{(2n-1)2n}{2}$, clearly $Z(n) \leq 2n-1$ for all n.

This inequality is best possible for even n, since $Z(2^k) = 2^{k+1} - 1$. We note that for odd n, we have $Z(n) \leq n - 1$, and this is best possible for odd n, since $Z(p) = p-1$ for prime p. By

$$\frac{Z(n)}{n} \leq 2 - \frac{1}{n} \quad \text{and} \quad \frac{Z(2^k)}{2^k} = 2 - \frac{1}{2^k}$$

we get $\limsup_{n \to \infty} \frac{Z(n)}{n} = 2.$ (3)

Since $Z\left(\frac{p(p+1)}{2}\right) = p$, and $\frac{p}{p(p+1)/2} \to 0$ $(p \to \infty)$, it follows

$$\liminf_{n \to \infty} \frac{Z(n)}{n} = 0 \tag{4}$$

For Z(Z(n)), the following can be proved. By

$Z\left(Z\left(\frac{p(p+1)}{2}\right)\right) = p-1$, clearly

$$\liminf_{n \to \infty} \frac{Z(Z(n))}{n} = 0 \tag{5}$$

On the other hand, by $Z(Z(n)) \leq 2Z(n) - 1$ and (3), we have

$$\limsup_{n \to \infty} \frac{Z(Z(n))}{n} \leq 4 \tag{6}$$

3. We now prove

$$\liminf_{n\to\infty} | Z(2n) - Z(n) | = 0 \qquad (7)$$

and

$$\limsup_{n\to\infty} | Z(2n) - Z(n) | = +\infty \qquad (8)$$

Indeed, in [1] it was proved that $Z(2p) = p-1$ for a prime $p \equiv 1 \pmod 4$. Since $Z(p) = p-1$, this proves relation (7).

On the other hand, let $n = 2^k$. Since $Z(2^k) = 2^{k+1} - 1$ and $Z(2^{k+1}) = 2^{k+2} - 1$, clearly $Z(2^{k+1}) - Z(2^k) = 2^{k+1} \to \infty$ as $k \to \infty$.

References

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