ON THE PSEUDO-SMARANDACHE SQUAREFREE FUNCTION

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Abstract. In this paper we discuss various problems and conjectures concered the pseudo-Smarandache squarefree function.

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For any positive integer n, the pseudo-Smarandache squarefree function $ZW(n)$ is defined as the least positive integer m such that m^n is divisible by n. In this paper we shall discuss various problems and conjectures concered $ZW(n)$.

1. The value of $ZW(n)$

By the definition of $ZW(n)$, we have $ZW(1)=1$. For $n>1$, we give a general result as follows.

Theoren 1.1. If $n>1$, then $ZW(n)=p_1p_2\cdots p_k$, where $p_1, p_2, \cdots, p_k$ are distinct prime divisors of n.

Proof. Let $m=ZW(n)$. Let $p_1, p_2, \cdots, p_k$ be distinct prime divisors of n. Since $n|m^n$, we get $p_i|m$ for $i=1, 2, \cdots, k$. It implies that $p_1p_2\cdots p_k|m$ and

(1.1) $$m\geqslant p_1p_2\cdots p_k.$$

On the other hand, let $r(i)$ $(i=1, 2, \cdots, k)$ denote the order of p_i $(i=1, 2, \cdots, k)$ in n. Then we have

(1.2) $$r(i) \le \frac{\log n}{\log p_i} < n, i = 1,2,\ldots,k.$$

Thus, we see from (1.2) that $(p_1p_2\cdots p_k)^n$ is divisible by n. It implies that

(1.3) $$m \leqslant p_1p_2\cdots p_k.$$

The combination of (1.1) and (1.3) yields $m= p_1p_2\cdots p_k$. The theorem is proved.

2. The difference $|ZW(n+1)-ZW(n)|$

In [3], Russo given the following two conjectures.

Conjecture 2.1. The difference $|ZW(n+1)-ZW(n)|$ is unbounded.

Conjecture 2.2. $ZW(n)$ is not a Lipschitz function.

In this respect, Russo [3] showed that if the Lehmer-Schinzel conjecture concered Fermat numbers is true (see [2]), then Conjectures 2.1 and 2.2 are true. However, the Lehmer-Schinzel conjecture is not resolved as yet. We now completely verify the above-mentioned conjectures as follows.

Theorem 2.1. The difference $|ZW(n+1)-ZW(n)|$ is unbounded.

Proof. Let p be an odd prime. Let $n=2^p-1$, and let q be a prime divisor of n. By a well known result of Birkhoff and Vandiver [1], we have $q=2lp+1$, where l is a positive integer. Therefore, by Theorem 1.1, we get

(2.1) $$ZW(n)=ZW(2^p-1) \geqslant q=2lp+1 \geqslant 2p+1.$$

On the other hand, apply Theorem 1.1 again, we get

(2.2) $$ZW(n+1)=ZW(2^p)=2.$$

By (2.1) and (2.2), we obtain

(2.3) $$|ZW(n+1)-ZW(n)|\geqslant 2p-1.$$

Notice that there exist infinitely many odd primes p. Thus, we find from (2.3) that the difference $|ZW(n+1)-ZW(n)|$ is unbounded. The theorem is proved.

As a direct consequence of Theorem 2.1, we obtain the following corollary.

Corollary 2.1. $ZW(n)$ is not a Lipschitz function.

3. The sum and product of the reciprocal of $ZW(n)$

Let **R** be the set of all real numbers. In [3], Russo posed the following two problems.

Problem 3.1. Evaluate the infinite product

(3.1) $$P=\prod_{n=1}^{\infty}\frac{1}{ZW(n)}.$$

Problem 3.2. Study the convergence of the infinite series

(3.2) $$S(a)=\sum_{n=1}^{\infty}\frac{1}{(ZW(n))^{a}}, a\in\mathbf{R}, a>0.$$

We now completely solve the above-mentioned problems as follows.

Theorem 3.1. $P=0$.

Proof. By Theorem 1.1, we get $ZW(n)>1$ for any positive integer n with $n>1$. Thus, by (3.1), we obtain $P=0$ immediately. The theorem is proved.

Theorem 3.2. For any positive number a, $S(a)$ is divergence.

Proof. we get from (3.1) that

$$S(a)=\sum_{n=1}^{\infty}\frac{1}{(ZW(n))^{a}}>\sum_{r=1}^{\infty}\frac{1}{(ZW(2^{r}))^{a}}. \tag{3.3}$$

By Theorem 1.1, we have

$$ZW(2^{r})=2\,, \tag{3.4}$$

for any positive integer r. Substitute (3.4) into (3.3), we get

$$S(a)=\sum_{r=1}^{\infty}\frac{1}{2^{r}}=\infty. \tag{3.5}$$

We find from (3.5) that $S(a)$ is divergence. The theorem is proved.

4. Diophantine equations concerning $ZW(n)$

Let **N** be the set of all positive integers. In [3], Russo posed the following problems concerned diophantine equations.

Problem 4.1. Find all solutions n of the equation

$$ZW(n)=ZW(n+1)ZW(n+2),\ n\in \mathbf{N}. \tag{4.1}$$

Problem 4.2. Solve the equation

$$ZW(n).\ ZW(n+1)=ZW(n+2),\ n\in \mathbf{N}. \tag{4.2}$$

Problem 4.3. Solve the equation

$$ZW(n).\ ZW(n+1)=ZW(n+2).\ ZW(n+3),\ n\in \mathbf{N}. \tag{4.3}$$

Problem 4.4. Solve the equation

$$ZW(mn)=m^{k}ZW(n),\ m,\ n,\ k\in \mathbf{N}. \tag{4.4}$$

Problem 4.5. Solve the equation

$$(ZW(n))^{k}=k.\ ZW(kn),\ k,\ n\in \mathbf{N},\ k>1,\ n>1. \tag{4.5}$$

Problem 4.6. Solve the equation

$$(ZW(n))^{k}+(ZW(n))^{k-1}+\cdots+ZW(n)=n,\ k,\ n\in \mathbf{N},\ k>1. \tag{4.6}$$

In this respect, Russo [3] showed that (4.1), (4.2) and (4.3) have no solutions n with $n \leqslant 1000$, and (4.6) has no solutions (n, k) satisfying $n \leqslant 1000$ and $k \leqslant 5$. We now completely solve the above-mentioned problems as follows.

Theorem 4.1. The equation (4.1) has no solutions n.

Proof. Let n be a solution of (4.1). Further let p be a prime divisor of $n+1$. By Theorem 1.1, we get $p|ZW(n+1)$. Therefore, by (4.1), we get $p|ZW(n)$. It implies that p is also a prime divisor of n. However, since gcd $(n, n+1)=1$, it is impossible. The theorem is proved.

By the same method as in the proof of Theorem 4.1, we can prove the following theorem without any difficult.

Theorem 4.2. The equation (4.2) has no solutions n.

Theorem 4.3. The equation (4.3) has no solutions n.

Proof. Let n be a solution of (4.3). Further let $p_1, p_2, \cdots, p_k$ and $q_1, q_2, \cdots, q_t$ be distinct prime divisors of $n(n+1)$ and $(n+2)(n+3)$ respectively. We may assume that

(4.7) $$p_1 < p_2 < \cdots < p_k,\ q_1 < q_2 < \cdots < q_t.$$

Since gcd $(n, n+1)$=gcd $(n+2, n+3)=1$, by Theorem 1.1, we get

$$ZW(n).\ ZW(n+1) = p_1 p_2 \cdots p_k$$

(4.8) $$ZW(n+2).\ ZW(n+3) = q_1 q_2 \cdots q_t$$

Substitute (4.8) into (4.3), we obtain

(4.9) $$p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_t.$$

By (4.7) and (4.9), we get $k=t$ and

(4.10) $$p_i = q_i,\ i=1, 2, \cdots, k.$$

Since gcd $(n+1, n+2)=1$, if $2|n$ and p_j $(1 \leqslant j \leqslant k)$ is a prime divisor of $n+1$, then from (4.10) we see that p_j is an odd prime with $p_j|n+3$.

Since gcd $(n+1, n+3)=1$ if $2| n$, it is impossible.

Similarly, if $2 \mid n$ and q_j $(i \leqslant j \leqslant k)$ is a prime divisor of $n+2$, then q_j is an odd prime with $q_j|n$. However, since $(n, n+2)=1$ if $2 \mid n$, it is impossible. Thus, (4.3) has no solutions n. The theorem is proved.

Theorem 4.4. The equation (4.4) has infinitely many solutions (m, n, k). Moreover, every solution (m, n, k) of (4.4) can be expressed as

$$m=p_1p_2\cdots p_r,\ n=t,\ k=1, \tag{4.11}$$

where $p_1, p_2, \cdots, p_r$ are distinct primes, t is a positive integer with gcd $(m, t)=1$.

Proof. Let (m, n, k) be a solution of (4.4). Further let $d=\gcd(m, n)$. By Theorem 1.1, we get from (4.4) that

$$ZW(mn)=ZW\left(\frac{m}{d}.n\right)=ZW\left(\frac{m}{d}\right).ZW(n)=m^kZW(n). \tag{4.12}$$

Since $ZW(n)\neq 0$, we obtain from (4.12) that

$$ZW\left(\frac{m}{d}\right)=m^k. \tag{4.13}$$

Furthermore, since $m \geqslant ZW(m)$, we see from (4.13) that $k=d=1$ and $m=p_1p_2\cdots p_r$, where $p_1, p_2, \cdots, p_r$ are distinct primes. Thus, the solution (m, n, k) can be expressed as (4.11). The theorem is proved.

Theorem 4.5. The equation (4.5) has infinitely many solutions (n, k). Moreover, every solution (n, k) of (4.5) can be expressed as

$$n=2^r,\ k=2,\ r\in \mathbf{N}. \tag{4.14}$$

Proof. Let (n, k) be a solution of (4.5). Further let $d=\gcd(n, k)$. By Theorem 1.1, we get from (4.5) that

$$ZW(nk)=kZW\left(n.\frac{k}{d}\right)=kZW(n).ZW\left(\frac{k}{d}\right)=(ZW(n))^k. \tag{4.15}$$

Since $ZW(n) \neq 0$ and $k > 1$, by (4.15), we obtain

$$kZW\left(\frac{k}{d}\right) = (ZW(n))^{k-1}. \tag{4.16}$$

Since $n > 1$, we find from (4.16) that k and n have the same prime divisors.

Let $p_1, p_2, \cdots, p_t$ be distinct prime divisors of n. Then we have $ZW(n) = p_1 p_2 \cdots p_t$. Since $ZW(k/d) \leqslant k$, we get from (4.16) that

$$k^2 \geq kZW\left(\frac{k}{d}\right) = (ZW(n))^{k-1} = (p_1 p_1 \cdots p_t)^{k-1}. \tag{4.17}$$

Since $k > 1$, by (4.17), we obtain $t=1$ and either

$$k=3, p_1=3, \tag{4.18}$$

or

$$k=2, p_1=2. \tag{4.19}$$

Recall that k and n have the same prime divisors. If (4.18) holds, then $ZW(k/d)=ZW(1)=1$ and (4,16) is impossible. If (4.19) holds, then the solution (n, k) can be expressed as (4.14). Thus, the Theorem is proved.

Theorem 4.6. The equation (4.6) has no solutions (n, k).

Proof. Let (n, k) be a solution of (4.6). Further let $m=ZW(n)$, and let $p_1, p_2, \cdots, p_t$ be distinct prime divisors of n. By Theorem 1.1, we have

$$n = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}, ZW(n) = p_1 p_2 \ldots p_t, \tag{4.20}$$

where $a_1, a_2, \cdots, a_t$ are positive integers. Substitute (4.20) into (4.6), we get

$$1 + p_1 p_2 \ldots p_t + \ldots + (p_1 p_2 \ldots p_t)^{k-1} = p_1^{a_1-1} p_2^{a_2-1} \ldots p_t^{a_t-1}. \tag{4.21}$$

Since gcd $(1, p_1 p_2 \cdots p_t)=1$, we find from (4.21) that $a_1=a_2=\cdots=a_t=1$. It implies that k=1, a contradiction. Thus, (4.6) has no solutions (n, k). The theorem is proved.

References

[1] Birkhoff, G. D. and Vandiver, H. S., On the integral divisor of a^n-b^n, Ann. of Math. (2), 1904, 5:173-180.

[2] Ribenboim, P., The book of prime numbers records, New York, Springer-Verleg, 1989.

[3] Russo, F., A set of new Smarandache functions, sequences and conjectures in number theory, Lupton, American Reserch Press, 2000.

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