ON THE QUATERNARY QUADRATIC DIOPHANTINE EQUATIONS

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In this paper are presented the parametric solutions for the homogeneous diophantine equations:
$x^2 + by^2 + cz^2 = w^2$ (1)
where b, c are rational integers.

I. Present theory. Case 1: $b = c = 1$
Curmichael [2] showed that the solutions are expresions with the form:

$w = p^2 + q^2 + u^2 + v^2$; $y = 2pq + 2uv$; (2)
$x = p^2 - q^2 + u^2 - v^2$; $z = 2pv - 2qu$;

where p, q, u, v are rational integers.
Mordell [3] showed that only these are the equations solution's by appying the arithmetic theory of the Gaussian integers.

Case 2: $b = 1$; $c = -1$. Mordell [3] showed that the solutions are, and only these, the expressions:

$2x = ad - bc$; $2y = ac + bd$; (3)
$2z = ac - bd$; $2w = ad + bc$;

a, b, c, d are integer parameters.

Case 3: b, c, are rational integers.

Mordell [3] took the particulary solutions with trhee parameters again, had been proposed by Euler:

$w = p^2 + bq^2 + cu^2$; $y = 2pq$; (4)
$x = p^2 - bq^2 - cu^2$; $z = 2pu$;

II. Results.
In [4] is proposed a new method to solve the quaternary equations using the notion of "quadratic combination". If we noted $G^2{}_2$, the complete system of equation's solutions: $x^2 + y^2 = z^2$, and also $G_3{}^2$ for the equation: $x^2 + y^2 + z^2 = w^2$, we sholl can to enuneiate:
Definition 1: Quadratic combination is a numerical function _ $\square$ _ which associates each two solutios from $G^2{}_2$, four solutions from $G_3{}^2$. Simbolicaly we have:

$\square: G^2{}_2 \times G^2{}_2 \rightarrow G_3{}^2$

Observation.

From the quadratic combination of the equation's solutions with the form: $x^2 + by^2 = z^2$, we sholl obtain the solutions for the equations $x^2 + by^2 + cz^2 = w^2$ [4]

1. Case b = c = 1

From the quadratic combination, we find again the solution (2). We can present another demonstration for Mordell's sentence. From [4] we have:

Theurema 1.

For the equation E_3^2, the solutions are expresions (2) and only these. The first part of the demonstration results by verification. For the ssecond part of it, we can use the property demonstrated in [4].

Lemma 2. The multitude of the equation's solutions E_3^2 is a graph F_3^2 as terminal top the ordinary solution (1, 0, 0, 1) and the arcs are given by the "t" functions:

$t = w \pm x \pm y \pm z$

The solutions are matriceally developed:

$$S_{i+1} = S_i \cdot B \quad \text{, with } B = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix} \qquad (5)$$

Lemma 3. Any solutions from the equations (2) are on the graph F_3^2 and, reciprocally, any solutions from the F_3^2 can be written with form (2).

It was defined the term: $t_1 = x + y + z - w$; $t_{i+1} < t_i$, where variables are naturale numbers [4]. We are verifing that form every solution of naturale numbers can derive a solution whit $w_1 < w$. The parameter's corespondence ($p > q$ and $u > v$) will be:

$$p_1 = p - q - v \; ; \quad u_1 = u + q - v \; ;$$
$$q_1 = q \quad ; \quad v_1 = v$$

It is obteinid a number of decreasing values w_1, having as limet the ordinary solutions (1, 0, 0, 1). Reciprocally, for every solution from the graph F_3^2 is obteined a number of parameterly solutions with w_1 breeder, in cas $t_{i+1} > t_i$.

2. Case b = 1 , c = -1. From quadratic combination resultes equations:

$$\begin{aligned} w &= p^2 + q^2 - u^2 - v^2 \\ x &= p^2 - q^2 + u^2 - v^2 \\ y &= 2pq + 2uv \\ z &= 2pv + 2qu \end{aligned} \qquad (6)$$

It can be showed that the Mordell's solutions (3) are equivalent with solutions (6); the parameter's equivalence is given by:

$a = p + v$; $b = p - v$

$c = q - u \qquad ; d = q + u$

3. Case b, c are rationale integers. For simplicity, we shell treat in two subcases:

3a) b, c prime numbers. The quadratic combination will require the solutions:

$$
\begin{aligned}
w &= p^2 + bq^2 + cu^2 + bcv^2 \\
x &= p^2 - bq^2 - cu^2 + bcv^2 \\
y &= 2pq + 2cuv \\
z &= 2pu - 2bqv
\end{aligned}
\qquad (7)
$$

3b) b and c are compound numbers. For any decomposition: $b = i \bullet j$ and $c = l \bullet h$, where i, j, l, h are rationale integers, we have the general solutions with four parameters of the equation (1):

$$
\begin{aligned}
w &= ihp^2 + jhq^2 + jlu^2 + ilv^2 \\
x &= ihp^2 - jhq^2 + jlu^2 - ilv^2 \\
y &= 2hpq + 2luv \\
z &= 2ipv - 2jqu
\end{aligned}
\qquad (8)
$$

III. Applications We sholl take again from [4] only the application of the numerical representations of exponent 2. It is well known the Fermont - Lagrange Theory.

Theorema 2

For any natural number it is at least a representation by sum of four whole number's square rest:

$$z = u^2 + v^2 + w^2 + t^2 \qquad (9)$$

Later on another Theory was demonstrated:

Theorema 3

For any natural number $z \neq 2^{2k}(8l + 7)$ it is least a representation of three whole a numbers:

$$z = u^2 + v^2 + w^2 \qquad (9')$$

Our theory allows us to enunciate a much stranger theory:

Theorema 4

For any natural number z it is at least three whole numbers (u, v, w) or (a, b, c), in order to have :

$$
\begin{aligned}
z &= u^2 + v^2 + w^2 \qquad (\alpha) \\
z &= a^2 + b^2 + 2c^2 \qquad (\beta)
\end{aligned}
\qquad (10)
$$

For $z = z_1 = 2^{2k}(8l + 7)$, we have only the representation (β), for $z = z_2 = 2^{2k+1}(8l + 7)$, we have only the representation (α) and for $z \neq z_1 \neq z_2$, we have in the same time the representations (α) and (β).

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