ON THE SIMPLE NUMBERS AND THE MEAN VALUE PROPERTIES*

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ABSTRACT. A number n is called simple number if the product of its proper divisors is less than or equal to n. In this paper, we study the mean value properties of the sequence of the simple numbers, and give several interesting asymptotic formulae.

1. INTRODUCTION

A number n is called simple number if the product of its proper divisors is less than or equal to n. For example: $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots$. In problem 23 of [1], Professor F.Smarandach asked us to study the properties of the sequence of the simple numbers. Let A is a set of simple numbers, that is, $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots\}$. In this paper, we use the elementary methods to study the properties of this sequence, and give several interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{1}{n} = (\ln\ln x)^2 + B_1 \ln\ln x + B_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where B_1, B_2 are the constants.

Theorem 2. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{1}{\phi(n)} = (\ln\ln x)^2 + C_1 \ln\ln x + C_2 + O\left(\frac{\ln\ln x}{\ln x}\right),.$$

where C_1, C_2 are the constants, $\phi(n)$ is Euler function.

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Theorem 3. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} \frac{1}{\sigma(n)} = (\ln\ln x)^2 + D_1 \ln\ln x + D_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where D_1, D_2 are the constants, $\sigma(n)$ is divisor function.

2. Some Lemmas

To complete the proof of the Theorems, we need the following two Lemmas: First Let n be a positive integer, $p_d(n)$ is the product of all positive divisors of n, that is, $p_d(n) = \prod_{d|n} d$. $q_d(n)$ is the product of all positive divisors of n but n, that is, $q_d(n) = \prod_{d|n, d<n} d$. Then we have

Lemma 1. *Let $n \in A$, then we have $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four cases.*

Proof. From the definition of $p_d(n)$ we know that

$$p_d(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So from this formula we have

$$p_d^2(n) = \prod_{d|n} d \times \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{d(n)}. \tag{1}$$

where $d(n) = \sum_{d|n} 1$. From (1) we immediately get $p_d(n) = n^{\frac{d(n)}{2}}$ and

$$q_d(n) = \prod_{d|n, d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}. \tag{2}$$

By the definition of the simple numbers and (2), we get $n^{\frac{d(n)}{2}-1} \leq n$. Therefor we have

$$d(n) \leq 4.$$

This inequality holds only for $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four cases. This completes the proof of Lemma 1.

Lemma 2. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{p \leq \sqrt{x}} \frac{1}{p} \ln\ln \frac{x}{p} = (\ln\ln x)^2 + B_1 \ln\ln x + B_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where B_1, B_2 are the constants.

Proof. It is clear that

$$\sum_{p\le\sqrt{x}} \frac{1}{p}\ln\ln\frac{x}{p} = \sum_{p\le\sqrt{x}} \frac{1}{p}\ln(\ln x-\ln p)$$

$$= \sum_{p\le\sqrt{x}} \frac{1}{p}\left(\ln\ln x+\ln\left(1-\frac{\ln p}{\ln x}\right)\right)$$

$$= \ln\ln x \sum_{p\le\sqrt{x}} \frac{1}{p} + \sum_{p\le\sqrt{x}} \frac{1}{p}\ln\left(1-\frac{\ln p}{\ln x}\right). \tag{3}$$

Applying

$$\sum_{p\le x}\frac{1}{p} = \ln\ln x + C_1 + O\left(\frac{1}{\ln x}\right). \tag{4}$$

we obtain

$$\ln\ln x \sum_{p\le\sqrt{x}}\frac{1}{p} = \ln\ln x\left(\ln\ln\sqrt{x}+C_1+O\left(\frac{1}{\ln x}\right)\right)$$

$$= (\ln\ln x)^2 + B_1\ln\ln x + O\left(\frac{\ln\ln x}{\ln x}\right). \tag{5}$$

If $m>2$, note that $\pi(x)=\frac{x}{\ln x}+\frac{x}{\ln^2 x}+O\left(\frac{x}{\ln^3 x}\right)$, then we have

$$\sum_{p\le\sqrt{x}}\frac{\ln^m p}{p} = \int_2^{\sqrt{x}}\frac{\ln^m y}{y}d\pi(y)$$

$$= \frac{\ln^m\sqrt{x}}{\sqrt{x}}\pi(\sqrt{x})+O(1)-\int_2^{\sqrt{x}}\pi(y)\frac{m\ln^{m-1}y-\ln^m y}{y^2}dy$$

$$= \frac{\ln^m\sqrt{x}}{\sqrt{x}}\left(\frac{\sqrt{x}}{\ln\sqrt{x}}+O\left(\frac{\sqrt{x}}{\ln^2\sqrt{x}}\right)\right)$$

$$-\int_2^{\sqrt{x}}\left(\frac{y}{\ln y}+\frac{y}{\ln^2 y}+O\left(\frac{y}{\ln^3 y}\right)\right)\frac{m\ln^{m-1}y-\ln^m y}{y^2}dy$$

$$= \frac{\ln^{m-1}x}{2^{m-1}}+O\left(\frac{\ln^{m-2}x}{2^{m-2}}\right)$$

$$+\int_2^{\sqrt{x}}\left[\frac{\ln^{m-1}y}{y}-(m-1)\frac{\ln^{m-2}y}{y}+O\left((1-m)\frac{\ln^{m-3}y}{y}\right)\right]dy$$

$$= \frac{\ln^{m-1}x}{2^{m-1}}+O\left(\frac{\ln^{m-2}x}{2^{m-2}}\right)+\frac{\ln^m x}{m2^m}-\frac{\ln^{m-1}x}{2^{m-1}}+O\left(\frac{(1-m)\ln^{m-2}x}{(m-2)2^{m-2}}\right)$$

$$= \frac{1}{m2^m}\ln^m x+O\left(\frac{1}{2^{m-2}(2-m)}\ln^{m-2}x\right). \tag{6}$$

From (6) and note that $\sum_{m=1}^{\infty}\frac{1}{m2^m}$ is convergent, we have

$$-\sum_{p\le\sqrt{x}}\frac{1}{p}\ln\left(1-\frac{\ln p}{\ln x}\right)$$

$$= \sum_{p\le\sqrt{x}} \frac{1}{p}\left(\frac{\ln p}{\ln x} + \frac{\ln^2 p}{2\ln^2 x} + \cdots + \frac{\ln^m p}{m\ln^m x} + \cdots\right)$$

$$= \frac{1}{\ln x}\sum_{p\le\sqrt{x}} \frac{\ln p}{p} + \frac{1}{2\ln^2 x}\sum_{p\le\sqrt{x}} \frac{\ln^2 p}{p} + \cdots + \frac{1}{m\ln^m x}\sum_{p\le\sqrt{x}} \frac{\ln^m p}{p} + \cdots$$

$$= \frac{1}{\ln x}\left(\frac{1}{2}\ln x + O(1)\right) + \cdots + \frac{1}{m\ln^m x}\left(\frac{1}{m2^m}\ln^m x + O\left(\frac{\ln^{m-2} x}{2^{m-2}(2-m)}\right)\right) + \cdots$$

(7)

$$= B_2 + O\left(\frac{1}{\ln x}\right),$$

where we have used the asymptotic formula $\sum_{p\le\sqrt{x}} \frac{\ln p}{p} = \frac{1}{2}\ln x + O(1)$ and the power series expansion $\ln(1-x) = -(x + \frac{x^2}{2} + \cdots + \frac{x^m}{m} + \cdots)$. From (3), (5) and (7) we immediately get

$$\sum_{p\le\sqrt{x}} \frac{1}{p}\ln\ln\frac{x}{p} = (\ln\ln x)^2 + B_1\ln\ln x + B_2 + O\left(\frac{\ln\ln x}{\ln x}\right).$$

This proves Lemma 2.

3. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. From Lemma 1 we have

$$\sum_{\substack{n\in A\\ n\le x}} \frac{1}{n} = \sum_{p\le x}\frac{1}{p} + \sum_{p^2\le x}\frac{1}{p^2} + \sum_{p^3\le x}\frac{1}{p^3} + \sum_{\substack{pq\le x\\ p\ne q}}\frac{1}{pq}$$

$$= \sum_{p\le x}\frac{1}{p} + \sum_{p^3\le x}\frac{1}{p^3} + \sum_{pq\le x}\frac{1}{pq}. \tag{8}$$

Applying (4) and Lemma 2 we get

$$\sum_{pq\le x}\frac{1}{pq} = 2\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q} - \left(\sum_{p\le\sqrt{x}}\frac{1}{p}\right)\left(\sum_{q\le\sqrt{x}}\frac{1}{q}\right)$$

$$= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\left(\ln\ln\frac{x}{p} + C_1 + O\left(\frac{1}{\ln x}\right)\right) - \left(\ln\ln\sqrt{x} + C_1 + O\left(\frac{1}{\ln x}\right)\right)^2$$

$$= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\ln\ln\frac{x}{p} + 2C_1\sum_{p\le\sqrt{x}}\frac{1}{p} + O\left(\frac{1}{\ln x}\sum_{p\le\sqrt{x}}\frac{1}{p}\right)$$

$$- \left((\ln\ln x)^2 + C_2\ln\ln x + C_3 + O\left(\frac{\ln\ln x}{\ln x}\right)\right)$$

$$= (\ln\ln x)^2 + C_4\ln\ln x + C_5 + O\left(\frac{\ln\ln x}{\ln x}\right). \tag{9}$$

Combining (4), (8) and (9) and note that $\sum_{p^3\le x}\frac{1}{p^3}$ is convergent, we immediately obtain

$$\sum_{\substack{n\in A\\ n\le x}}\frac{1}{n}=(\ln\ln x)^2+B_1\ln\ln x+B_2+O\left(\frac{\ln\ln x}{\ln x}\right).$$

This completes the proof of Theorem 1.

Now we complete the proof of Theorem 2 and Theorem 3. From the definitions and the properties of Euler function and divisor function, and applying Lemma 1 we have

$$\sum_{\substack{n\in A\\ n\le x}}\frac{1}{\phi(n)}=\sum_{p\le x}\frac{1}{p-1}+\sum_{p^2\le x}\frac{1}{p^2-p}+\sum_{p^3\le x}\frac{1}{p^3-p^2}+\sum_{\substack{pq\le x\\ p\ne q}}\frac{1}{(p-1)(q-1)}$$

and

$$\sum_{\substack{n\in A\\ n\le x}}\frac{1}{\sigma(n)}=\sum_{p\le x}\frac{1}{p+1}+\sum_{p^2\le x}\frac{1}{p^2+p+1}+\sum_{p^3\le x}\frac{1}{p^3+p^2+p+1}+\sum_{\substack{pq\le x\\ p\ne q}}\frac{1}{(p+1)(q+1)}.$$

Note that $\frac{1}{p\pm1}=\frac{1}{p}\mp\frac{1}{p(p\pm1)}$ and $\sum_{p}\frac{1}{p(p\pm1)}$ is convergent, then using the methods of proving Theorem 1 we can easily deduce that

$$\sum_{\substack{n\in A\\ n\le x}}\frac{1}{\phi(n)}=(\ln\ln x)^2+C_1\ln\ln x+C_2+O\left(\frac{\ln\ln x}{\ln x}\right)$$

and

$$\sum_{\substack{n\in A\\ n\le x}}\frac{1}{\sigma(n)}=(\ln\ln x)^2+D_1\ln\ln x+D_2+O\left(\frac{\ln\ln x}{\ln x}\right).$$

This completes the proof of the Theorems.

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