ON THE SMARANDACHE DOUBLE FACTORIAL FUNCTION

Maohua Le

Abstract. In this paper we discuss various problems and conjectures concerned the Smarandache double factorial function.

Keywords: Smarandache double factorial function, inequality, infinite series, infinite product, diophantine equation

For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer m such that $m!!$ is divisible by n, where

$$m!!=\begin{cases} 2.4\ldots m, & \text{if} \quad 2\,|\,m, \\ 1.3\ldots m, & \text{if} \quad 2\,|\,m. \end{cases}$$

In this paper we shall discuss various problems and conjectures concerned $Sdf(n)$.

1. The valua of $Sdf(n)$

By the definition of $Sdf(n)$, we have $Sdf(1)=1$ and $Sdf(n)>1$ if $n>1$. We now give three general results as follows.

Theorem 1.1. If $2\nmid n$ and

$$n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \tag{1.1}$$

is the factorization of n, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $a_1, a_2, \cdots, a_k$ are positive integers, then

(1.2) $$Sdf(n)=\max\left(Sdf\left(p_1^{a_1}\right),Sdf\left(p_2^{a_2}\right),\ldots,Sdf\left(p_k^{a_k}\right)\right)$$

Proof. Let $m_i = sdf\left(P_i^{a_i}\right)$ for i=1, 2, ⋯, k. Then we get $2 \nmid m_i$ (i=1, 2, ⋯, k) and

(1.3) $$p_i^{a_i} \mid m_i!!, i=1,2,\ldots,k.$$

Furthur let m=max(m_1, m_2, ⋯, m_k). Then we have

(1.4) $$m_i!!|m!!,\ i=1,2,\cdots,k.$$

Therefore, by (1.3) and (1.4), we get

(1.5) $$p_i^{a_i} \mid m!!, i=1,2,\ldots,k.$$

Notice that p_1, p_2, ⋯, p_k are distinct odd primes. We have

(1.6) $$\gcd(p_i^{a_i}, p_j^{a_j})=1, 1 \le i < j \le k.$$

Thus, by (1.1), (1.5) and (1.6), we obtain $n|m!!$. It implies that

(1.7) $$Sdf(n) \leqslant m.$$

On the other hand, by the definition of m, if $Sdf(n) < m$, then there exists a prime power $p_j^{a_j}(1 \le j \le k)$ such that

(1.8) $$p_j^{a_j} \mid Sdf(n)!!.$$

By (1.1) and (1.8), we get $n \mid Sdf(n)!!$, a contradiction. Therefore, by (1.7), we obtain $Sdf(n)=m$. It implies that (1.2) holds. The theorem is proved.

Theorem 1.2. If $2|n$ and

(1.9) $$n=2^a n_1,$$

where a, n_1 are positive integers with $2 \nmid n_1$, then

(1.10) $$Sdf(n) \leqslant \max(Sdf(2^a), 2Sdf(n_1)).$$

Proof. Let m_0=$Sdf(2^a)$ and m_1=$Sdf(n_1)$. Then we have

(1.11) $$2^a | m_0!!,\ n_1 | m_1!!.$$

Since $(2m_1)!!=2.4\cdots(2m_1)=2^{m_1}\cdot m_1!=2^{m_1}\cdot m_1!!(m_1-1)!!$, we get $m_1!!|(2m_1)!!$. It implies that

(1.12) $$n_1|(2m_1)!!.$$

Let $m=\max(m_0, 2m_1)$. Then we have $m_0!!|m!!$ and $(2m_1)!!|m!!$. Since gcd $(2^a, n_1)=1$, we see from (1.9), (1.11) and (1.12) that $n|m!!$. Thus, we obtain $Sdf(n) \leqslant m$. It implies that (1.10) holds. The theorem is proved.

Theorem 1.3. Let a, b be two positive integers. Then we have

(1.13) $$Sdf(ab) \le \begin{cases} Sdf(a)+Sdf(b), & \text{if } 2|a \text{ and } 2|b, \\ Sdf(a)+2Sdf(b), & \text{if } 2|a \text{ and } 2|b, \\ 2Sdf(a)+2Sdf(b)-1, & \text{if } 2|a \text{ and } 2|b. \end{cases}$$

Proof. By Theorem 4.13 of [4], if $2|a$ and $2|b$, then

(1.14) $$Sdf(a)=2r,\ Sdf(b)=2s,$$

where r, s are positive integers. We see from (1.14) that

(1.15) $$a|(2r)!!,\ b|(2s)!!.$$

Notice that

(1.16) $$\frac{(2r+2s)!!}{(2r)!!(2s)!!}=\frac{2^{r+s}\cdot (r+s)!}{(2^r\cdot r!)(2^s\cdot s!)}=\frac{(r+s)!}{r!s!}=\binom{r+s}{r},$$

where $\binom{r+s}{r}$ is a binomial coefficient. Since $\binom{r+s}{r}$ is a positive integer, we see from (1.16) that

(1.17) $$(2r)!!(2s)!!|(2r+2s)!!$$

Thus, by (1.15) and (1.17), we get $ab|(2r+2s)!!$. It implies that

(1.18) $$Sdf(ab)\leqslant 2r+2s, \text{ if } 2|a \text{ and } 2|b.$$

If $2|a$ and $2\nmid b$, then

(1.19) $$Sdf(a)=2r,\ Sdf(b)=2s+1,$$

where a is a positive integer and s is a nonnegative integer. By (1.19), we get

(1.20) $$a|(2r)!!,\ b|(2s+1)!!.$$

Notice that

(1.21) $$\frac{(2r+4s+2)!!}{(2r)!!(2s+1)!!}=\frac{2^{r+2s+1}\cdot (r+2s+1)!}{2^{r}\cdot r!}\cdot\frac{2^{s}\cdot s!}{(2S+1)!}$$
$$=2^{3s+1}\cdot s!\frac{(r+2s+1)!}{r!(2s+1)}=2^{3s-1}\cdot s!\binom{r+2s+1}{r}.$$

We find from (1.21) that

(1.22) $$(2r)!!(2s+1)!!|(2r+4s+2)!!.$$

Thus, by (1.20) and (1.22), we obtain $ab|(2r+4s+2)!!$. It implies that

(1.23) $$Sdf(ab)\leqslant 2r+4s+2, \text{ if } 2|a \text{ and } 2|b.$$

If $2\mid a$ and $2\mid b$, then

(1.24) $$Sdf(a)=2r+1,\ Sdf(b)=2s+1,$$

where r, s are nonnegative integers. By (1.24), we get

(1.25) $$a|(2r+1)!!,\ b|(2s+1)!!.$$

Notice that

$$(1.26)\quad \frac{(4r+4s+3)!!}{(2r+1)!!(2s+1)!!}=\frac{(4r+4s+3)!}{(4r+4s+2)!!}\cdot\frac{(2r)!!}{(2r+1)!}\cdot\frac{(2s)!!}{(2s+1)!}$$

$$=\frac{(4r+4s+3)!}{2^{2r+2s+1}\cdot(2r+2s+1)!}\cdot\frac{2^{r}\cdot r!}{(2r+1)!}\cdot\frac{2^{s}\cdot s!}{(2s+1)!}$$

$$=\frac{r!s!}{2^{r+s+1}}\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1},$$

where $\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1}$ is a polynomial coefficient. Since $\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1}$ is a positive integer and $(2r+1)!!$, $(2s+1)!!$ are odd integers, we see from (1.26) that

$$(1.27)\qquad (2r+1)!!(2s+1)!!|(4r+4s+3)!!.$$

Thus, by (1.25) and (1.27), we get $ab|(4r+4s+3)!!$. It implies that

$$(1.28)\qquad Sdf(ab)\leqslant 4r+4s+3, \text{ if } 2\mid a \text{ and } 2\mid b.$$

The combination of (1.18), (1.23) and (1.28) yields (1.13). The n theorem is proved.

Theorem 1.4 Let p be a prime and let a be a positive integer. The we have

$$(1.29)\qquad p|Sdf(p^{a}).$$

Proof. Let $m=Sdf(p^{a})$. By Theorem 4.13 of [4], if p=2, then m is even. Hence, (1.29) holds for p=2. If $p>2$, then m is an odd integer with

$$(1.30)\qquad p^{a}|m!!$$

We now suppose that $p \mid m$. Let t be the greatest odd integer such that $t < m$ and $p|t$. Then we have

(1.31) $$m!!=t!!(t+2)\cdots(m-2)m,$$

where $t+2, \cdots, m-2, m$ are integers satisfying $p \nmid (t+2)\cdots(m-2)m$. Therefore, by (1.30) and (1.31), we get

(1.32) $$p^{a}|t!!$$

By (1.32), we get $m=Sdf(p^{q}) \leqslant t < m$, a contradiction. Thus, we obtain $p|m$. The theorem is proved.

Theorem 1.5 Let p be the least prime divisor of n. Then we have

(1.33) $$Sdf(n) \geqslant p.$$

Proof. Let $m=Sdf(n)$. By Theorem 4.13 of [4], if $2|n$, then $p=2$ and m is an even integer. So we get (1.33).

If $2 \mid n$, let $n=p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $a_1, a_2, \cdots, a_k$ are positive integers. By Theorem 1.1, we get

(1.34) $$m=\max\left(Sdf\left(p_1^{a_1}\right), Sdf\left(p_2^{a_2}\right), \ldots, Sdf\left(p_k^{a_k}\right)\right)$$

Further, by Theorem 1.4, we have $p_i \mid Sdf\left(p_i^{a_i}\right)$ for $i=1, 2, \cdots, k$. It implies that $Sdf\left(p_i^{a_i}\right) \geq p_i$ for $i=1, 2, \cdots, k$. Thus, by (1.34), we obtain

(1.35) $$m \geqslant \min(p_1, p_2, \cdots, p_k)=p.$$

The theorem is proved.

Theorem 1.6 For any positive integer n, we have

(1.36) $$Sdf(n!)=\begin{cases} n, & \text{if} \quad n=1,2, \\ 2n, & \text{if} \quad n>2. \end{cases}$$

Proof. Let $m=Sdf(n!)$. Then (1.36) holds for $n=1, 2$. If $n>2$, then both $n!$ and m are even. Since $(2n)!!=2^n n!$, we get

(1.37) $$m \leqslant 2n.$$

If $m<2n$, then $m=2n-2r$, where r is a positive integer. Since $m=Sdf(n!)$,

(1.38) $$\frac{(2n-2r)!!}{n!}=\frac{2^{n-r}\cdot(n-r)!}{n!}=\frac{2^{n-r}}{(n-r+1)\ldots(n-1)n}$$

must be an integer. But, since either n-1 or n is an odd integer great than 1, it is impossible by (1.38). Thus, by (1.37), we obtain $m=2n$. The theorem is proved.

Theorem 1.7 The equality

(1.39) $$Sdf(n)=n$$

holds if and only if n satisfies one of the following conditions:

(i) $n=1, 9$.

(ii) $n=p$, where p is a prime.

(iii) $n=2p$, where p is a prime.

Proof. Let $m=Sdf(n)$. If $2 \mid n$, let $n=p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$ be the factorization of n. By Theorem 1.1, we (1.34). Further, by Theorem 4.7 of [4], we have

(1.40) $$Sdf\left(p_i^{a_i}\right)\leq p_i^{a_i}, i=1,2,\ldots,k.$$

Therefore, by (1.34) and (1.40), we obtain

(1.41) $$m\leq \max\left(p_1^{a_1},p_2^{a_2},\ldots,p_k^{a_k}\right)$$

It implies that if $k>1$, then $m<n$. If $k=1$ and (1.39) holds, then

(1.42) $$m = Sdf\left(p_1^{a_1}\right) = p_1^{a_1}.$$

By Theorem 4.1 of [4], (1.42) holds for a_1=1. Since 2 | n, p_1 is an odd prime. By Theorem 1.3, if (1.42) holds, then we have

(1.43) $$p_1^{a_1} = m = Sdf\left(p_1^{a_1}\right) = Sdf(p_1 p_1 \ldots p_1) \leq 2a_1 \cdot Sdf(p_1) - 1 = 2a_1 p_1 - 1$$

Since $p_1 \geqslant 3$, (1.43) is impossible for $a_1 > 2$. If a_1=2, then from (1.43) we get

(1.44) $$p_1^2 \leqslant 4p_1 - 1,$$

whence we obtain p_1=3. Thus, (1.39) holds for an odd integer n if and only if n=1.9 or p, where p is an odd prime.

If 2|n, then n can be rewritten as (1.9), where n_1 is an odd integer with $n_1 \geqslant 1$. By Theorem 1.2, if (1.39) holds, then we have

(1.45) $$n = 2^a n_1 \leqslant \max(Sdf(2^a), 2Sdf(n_1)).$$

We see from (1.45) that if (1.39) holds, then either n_1=1 or a=1.

When n_1=1, we get from (1.39) that a=1 or 2. When a=1, we get,

(1.46) $$2n_1 = Sdf(2n_1).$$

It is a well known fact that if n_1 is not an odd prime, then there exists a positive integer t such that $t < n_1$ and $n_1 | t!$. Since $(2t)!! = 2^t \cdot t!$, we get

(1.47) $$Sdf(2n_1) \leqslant 2t \leqslant 2n_1,$$

a contradiction. Therefore, n_1 must be an odd prime. In this case, if $Sdf(2n_1) < 2n_1$, then $Sdf(2n_1) = 2n_1 - 2r$, where r is a positive integer. But, since

(1.48) $$\frac{(2n_1 - 2r)!!}{2n_1} = \frac{2^{n_1 - r} \cdot (n_1 - r)}{2n_1} = \frac{2^{n_1 - r - 1} \cdot (n_1 - r)!}{n_1}$$

is not an integer, it is impossible. Thus, (1.39) holds for an even integer if and only if $n=2p$, where p is a prime. The theorem is proved.

2. The inequalities concerned $Sdf(n)$

Let n be a positive integer. In [4], Russo posed the following problems and conjectures.

(2.1) $$\frac{n}{Sdf(n)} \leq \frac{n}{8} + 2$$

Problem 2.1. Is the inequality
true for any n?

Problem 2.2. Is the inequality

(2.2) $$\frac{Sdf(n)}{n} > \frac{1}{n^{0.73}}$$

true for any n?

Problem 2.3. Is the inequality

(2.3) $$\frac{1}{n \cdot Sdf(n)} < n^{-5/4}$$

true for any n?

(2.4) $$\frac{1}{n} + \frac{1}{Sdf(n)} < n^{-1/4}$$

Problem 2.4. Is the inequality
true for any n with $n > 2$?

Conjecture 2.1. For any positive number ε, there exist some n such that

(2.5) $$\frac{Sdf(n)}{n} < \varepsilon$$

In this respect, Russo [4] showed that if $n \leqslant 1000$, then the inequalities (2.1), (2.2), (2.3) and (2.4) are true. We now completely solve the above-mentioned questions as follows.

Theorem 2.1. For any positive integer n, the inequality (2.1) is true.

Proof. We may assume that $n > 1000$. Since $m!! \leqslant 945$ for $m=1, 2, \cdots, 9$, if $n > 1000$, then $Sdf(n) \geqslant 10$. So we have

(2.6) $$\frac{n}{Sdf(n)} \leq \frac{n}{10} < \frac{n}{8} + 2.$$

It implies that (2.1) holds. The theorem is proved.

The above theorem shows that the answer of Problem 2.1 is "yes".

In order to solve Problems 2.2, 2.3 and 2.4, we introduce the following result.

Theorem 2.2. If $n=(2r)!!$, where r is a positive integer withr $r \geqslant 20$, then

(2.7) $$Sdf(n) < n^{0.1}.$$

Proof. We now suppose that

(2.8) $$Sdf(n) \geqslant n^{0.1}.$$

Since $n=(2r)!!$, we get $Sdf(n)=2r$. Substitute it into (2.8), we obtain that if $r \geqslant 20$, then

(2.9) $$2r \geqslant ((2r)!!)^{0.1} = 2^{0.1r}(r!)^{0.1} \geqslant 2^2(r!)^{0.1}.$$

By the Strling theorem (see [1]), we have

(2.10) $$r! > \sqrt{2\pi r}\left(\frac{r}{e}\right)^r.$$

Since $r \geqslant 20$, we get $r/e > \sqrt{r}$. Hence, by (2.9) and (2.10), we obtain

(2.11) $$2r \geqslant 4(r!)^{0.1} > 4r^{0.05r} \geqslant 4r,$$

a contradiction. Thus, we get (2.7). The theorem is proved.

By the above theorem, we obtain the following corollary immediately.

Corollary 2.1. If $n=(2r)!!$, where r is a positive integer with $r \geqslant 20$, then the inequalities (2.2), (2.3) and (2.4) are false.

The above corollary means that the answers of Problems 2.2, 2.3 and 2.4 are "no".

Theorem 2.3. For any positive number ε, there exist some positive integers n satisfy (2.5).

Proof. Let $n=(2r)!!$, where r is a positive integer with $r \geqslant 20$. By Theorem 2.2, we have

(2.12) $$\frac{Sdf(n)}{n} < \frac{n^{0.1}}{n} = \frac{1}{n^{0.9}}.$$

By (2.12), we get

(2.13) $$\lim_{r \to \infty} \frac{Sdf(n)}{n} = 0.$$

Thus, by (2.13), the theorem is proved.

By the above theorem, we see that Conjecture 2.1 is true.

3. The difference $|Sdf(n+1)-Sdf(n)|$

In [4], Russo posed the following problem.

Problem 3.1. Is the difference $|Sdf(n+1)-Sdf(n)|$ bounded or unbounded?

We now solve this problem as follows.

Theorem 3.1. The difference $|Sdf(n+1)-Sdf(n)|$ is unbounded.

Proof. Let m be a positive integer, and let p be a prime. Further let ord $(p, m!)$ denote the order of p in m. For any positive integer a, it is a well known fact that

(3.1) $$\operatorname{ord}(p,a!)=\sum_{k=1}^{\infty}\left[\frac{a}{pk}\right].$$

(see Theorem 1.11.1 of [3]).

Let r be a positive integer. Then we have

(3.2) $$2^r!!=2\cdot 4\cdots 2^r=2^s\cdot 2^{r-1}!,$$

where

(3.3) $$s=2^{r-1}.$$

By (3.1), (3.2) and (3.3), we get

(3.4) $$\operatorname{ord}(2, 2^r!!)=2^{r-1}+\operatorname{ord}(2, 2^{r-1}!)=2^{r-1}+(2^{r-2}+\cdots+2+1)=2^r-1$$

Let $n=2^t$, where $t=2^r$. Then, by (3.4), we get

(3.5) $$Sdf(n)=2^r+2$$

On the other hand, then $n+1=2^t+1$ is a Fermat number. By the proof of Theorem 5.12.1 of [3], every prime divisor q of $n+1$ is the form $q=2^{r+1}l+1$, where l is a positive integer. It implies that

(3.6) $$q\geqslant 2^{r+1}+1.$$

Since $n+1$ is an odd integer, by Theorem 1.4, we get from (3.6) that

(3.7) $$Sdf(n+1)\geqslant q\geqslant 2^{r+1}+1.$$

We see from (3.8) that the difference $|Sdf(n+1)-Sdf(n)|$ is unbounded. Thus, the theorem is proved.

4. Some infinite series and products concerned *Sdf*(*n*)

In [4], Russo posed the following problems.

Problem 4.1. Evaluate the infinite series

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{Sdf(n)}. \tag{4.1}$$

Problem 4.2. Evaluate the infinite product

$$P=\prod_{n=1}^{\infty}\frac{1}{Sdf(n)}. \tag{4.2}$$

We now solve the above-mentioned problems as follows.

Theorem 4.1. $S=\infty$.

Proof. For any nonnegative integer m, let

$$g(m)=\frac{-1}{Sdf(2m+1)}+\sum_{i=1}^{\infty}\frac{1}{Sdf\left(2^i(2m+1)\right)}. \tag{4.3}$$

By (4.1) and (4.3), we get

$$S=\sum_{m=0}^{\infty}g(m). \tag{4.4}$$

We see from (4.3) that

$$g(0)=-1+\frac{1}{Sdf(2)}+\frac{1}{Sdf(4)}+\frac{1}{Sdf(8)}+\ldots \tag{4.5}$$

$$=-1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\ldots>\frac{1}{6}.$$

For positive integer m, let $t=Sdf(2m+1)$. Then t is an odd integer with $t \geqslant 3$. Notice that $2m+1 | t!!$ and

$$(2t)!!=2^t \cdot t!!. \tag{4.6}$$

We get from (4.6) that $2^j(2m+1)|(2t)!!$ for $j=1, 2, \cdots, t$. It implies that

(4.7) $$Sdf(2^j(2m+1))\leqslant 2t, j=1, 2, \cdots, t.$$

Therefore, by (4.3) and (4.7), we obtain

(4.8) $$g(m)>-\frac{1}{t}+\frac{1}{2t}+\frac{1}{2t}+\frac{1}{2t}=\frac{1}{2t}.$$

On the other hand, by Theorem 4.7 of [4], we have $t\leqslant 2m+1$. By (4.8), we get

(4.9) $$g(m)>\frac{1}{2(2m+1)}.$$

Thus, by (4.4), (4.5) and (4.9), we obtain

(4.10) $$S>\frac{1}{6}+\sum_{m=1}^{\infty}\frac{1}{2(2m+1)}=\infty.$$

The theorem is proved.

Theorem 4.2. $P=0$.

Proof. Since $Sdf(n)>1$ if $n>1$, by (4.2), we get $p=0$ immediately. The theorem is proved.

5. The diophantine equations concerned *Sdf(n)*

Let **N** be the set of all positive integers. In [4], Russo posed the following problems.

Problem 5.1 Find all the solutions n of the equation

(5.1) $$Sdf(n)!=Sdf(n!),\ n\in \mathbf{N}.$$

Problem 5.2 Is the equation

(5.2) $$(Sdf(n))^k=k\cdot Sdf(nk),\ n, k\in \mathbf{N}, n>1, k>1$$

have solutions (n, k)?

Problem 5.3 Is the equation

(5.3) $$Sdf(mn)=m^k \cdot Sdf(m),\ m, n, k \in \mathbf{N}$$

have solutions (m, n, k)?

We now completely solve the above-mentioned problems as follows.

Theorem 5.1 The equation (5.1) has only the solutions $n=1, 2, 3$.

Proof. Clearly, (5.1) has solutions $n=1, 2, 3$. We suppose that (5.1) has a solution n with $n>3$. By Theorem 1.6, if $n>2$, then

(5.4) $$Sdf(n)!=2n.$$

Substitute (5.4) into (5.1), we get

(5.5) $$Sdf(n)!=2n.$$

Let $m=Sdf(n)$. If $n>3$ and $2 \mid n$, then $n \geqslant 5$, $m \geqslant 5$ and $4|m!$. However, since $2||2n$, (5.5) is impossible.

If $n>3$ and $2|n$, then $m=2t$, where t is a positive integer with $t>1$. From (5.5), we get

(5.6) $$(2t)!=2n.$$

Since $m=Sdf(n)$, we have $n|(2t)!!$. It implies that

$$\frac{(2t)!!}{n}=\frac{2\cdot(2t)!!}{(2t)!}=\frac{2\cdot(2t)!!}{(2t)!!(2t-1)!!}=\frac{2}{(2t-1)!!}$$

must be an integer. But, since $t>1$, it is impossible. Thus, (5.1) has no solutions n with $n>3$. The theorem is proved.

Theorem 5.2 The equation (5.2) has only the solutions $(n, k)=(2, 4)$ and $(3, 3)$.

Proof. Let (n, k) be a solution of (5.2). Further, let $m=Sdf(n)$. By Theorem 1.3, we get

(5.7) $$Sdf(nk) < 2 \cdot Sdf(n) + 2 \cdot Sdf(k) \geq 2(m+k).$$

Hence, by (5.2) and (5.7), we obtain

(5.8) $$m^k < 2k(m+k),\ m > 1,\ k > 1.$$

If $m=2$, then from (5.8) we get $k \leqslant 6$. Notice that $n=2$ if $m=2$. We find from (5.2) that if $m=2$ and $k \leqslant 6$, then (5.2) has only the solution $(n, k)=(2, 4)$

If $m=3$, then from (5.8) we get $k \leqslant 3$. Since $n=3$ if $m=3$. We see from (5.2) that if $m=2$ and $k \leqslant 3$, then (5.2) has only the solution $(n, k)=(3, 3)$

If $m=4$, then from (5.8) we get $k \leqslant 2$. Notice that $n=4$ or 8 if $m=4$ and $n=5$ or 15 if $m=5$. Then (5.2) has no solution (n, k). Thus, (5.2) has only the solutions $(n, k)=(2, 4)$ and (3.3). The theorem is proved.

Theorem 5.3. All the solutions (m, n, k) of (5.3) are given in the following four classes:

(i) $m=1$, n and k are positive integers.

(ii) $n=1$, $k=1$, $m=1, 9, p$ or 2p, where p is a prime.

(iii) $m=2$, $k=1$, n is 2 or an odd integer with $n \geq 1$.

(iv) $m=3$, $k=1$, $n=3$.

Proof. If $m=1$, then (5.3) holds for any positive integers n and k. By Theorem 1.7, if $n=1$, then from (5.3) we get (ii). Thus, (i) and (ii) are proved.

Let (m, n, k) be a solution of (5.3) satisfying $m > 1$ and $n > 1$. By Theorem 1.3, if $2|m$ and $2|n$, then we have

(5.9) $$Sdf(mn) \leqslant Sdf(m)+Sdf(n).$$

Further, by Theorem 4.7 of [4], $Sdf(m) \leqslant m$. Therefore, by (5.3) and (5.9), we obtain

(5.10) $$m \geqslant (m^k-1)Sdf(n).$$

When n=2, we get from (5.10) that m=2 and k=1.

When $n>2$, we get $Sdf(n) \geqslant 4$ and (5.10) is impossible.

If $2|m$ and $2 \mid n$, then

(5.11) $$Sdf(mn) \leqslant Sdf(m)+2 \cdot Sdf(n).$$

Notice that $m \geqslant 2$, n is an odd integer with $n \geqslant 3$, $Sdf(n) \geqslant 3$. We obtain from (5.3) and (5.11) that

(5.12) $$m \geqslant Sdf(m) \geqslant (m^k-2)Sdf(n) \geqslant 3(m^k-2) \geqslant 3(m-2).$$

From (5.12), we get m=2. Then, by (5.3), we obtain

(5.13) $$Sdf(2n)=2^k \cdot Sdf(n).$$

Since $Sdf(2n) \leqslant 2n$, we see from (5.13) that k=1 and

(5.14) $$Sdf(2n)=2 \cdot Sdf(n).$$

Notice that (5.14) holds for any odd integer n with $n \geqslant 1$. We get (iii).

If $2 \mid m$ and $2|n$, then we have

(5.15) $$Sdf(mn) \leqslant 2 \cdot Sdf(m)+Sdf(n).$$

By (5.3) and (5.15), we get

(5.16) $$2m \geqslant 2 \cdot Sdf(m) \geqslant (m^k-1) \cdot Sdf(n).$$

When n=2, we see from (5.16) that m=3 and k=1. When $n>2$, we get from (5.16) that $2m \geqslant 4(m^k-1) \geqslant 4(m-1) > 2m$, a contradiction.

If $2 \mid m$ and $2 \mid n$, then we have

(5.17) $$Sdf(mn) \leqslant 2 \cdot Sdf(m)+2 \cdot Sdf(n)-1.$$

By (5.3) and (5.17), we get

(5.18) $$2m-1 \geqslant 2 \cdot Sdf(m)-1 \geqslant (m^k-2) \cdot Sdf(n) \geqslant 3(m^k-2).$$

It implies that k=1 and m=3 or 5. Wher m=3 and k=1, we get from (5.3) that

(5.19) $$Sdf(3n)=3 \cdot Sdf(n).$$

Since $Sdf(3n) \leqslant Sdf(n)+6$, we find from (5.19) that n=3. Thus, we get (iv). When m=5 and k=1, we have

(5.20) $$Sdf(5n)=5 \cdot Sdf(n).$$

Since $Sdf(5n) \leqslant Sdf(n)+10$, (5.20) is impossible. To sum up, the theorem is proved.

Let p be a prime, and let $N(p)$ denote the number of solutions x of the equation

(5.21) $$Sdf(x)=p, x \in \mathbf{N}.$$

Recently, Johnson showed that if p is an odd prime, then

(5.22) $$N(p)=2^{(p-3)/2}.$$

Unfortunately, the above-mentioned result is false. For example, by (5.22), we get $N(19)=2^8=256$. However, the fact is that $N(19)=240$. We now give a general result as follows.

Theorem 5.4. For any positive integer t, let $p(t)$ denote the tth odd prime. If $p=p(t)$, then

$$(5.23) \qquad N(p)=\prod_{i=1}^{t-1}(a(i)+1),$$

where

$$(5.24) \quad a(i)=\sum_{m=1}^{\infty}\left(\left[\frac{p-2}{(p(i))^m}\right]-\left[\frac{(p-3)/2}{(p(i))^m}\right]\right), i=1,2,\ldots,t-1.$$

Proof. Let x be a solution of (5.21). It is an obvious fact that

$$(5.25) \qquad x=dp.$$

where d is a divisor of $(p\text{-}2)!!$. So we have

$$(5.26) \qquad N(p)=d((p\text{-}2)!!),$$

where $d((p\text{-}2)!!)$ is the number of distinct divisors d of $(p\text{-}2)!!$.

By the definition of $(p\text{-}2)!!$, we have

$$(5.27) \qquad (p\text{-}2)!!=(p(1))^{a(1)}(p(2))^{a(2)}\cdots(p(t\text{-}1))^{a(t\text{-}1)},$$

where

$$(5.28) \qquad a(i)=\mathrm{ord}(p(i), (p\text{-}2)!!),\ i=1, 2, \cdots, t\text{-}1.$$

Notice thet

$$(5.29) \qquad (p-2)!!=\frac{(p-2)!}{2^{(p-3)/2}\cdot\left(\frac{p-3}{2}\right)!}.$$

We get

$$(5.30) \mathrm{ord}(p(i),(p-2)!!)=\mathrm{ord}(p(i),(p-2)!)-\mathrm{ord}\left(p(i),\left(\frac{p-3}{2}\right)!\right),$$

Therefore, by Theorem 1.11.1 of [3], we see from (5.28) and (5.30) that $a(i)$ (i=1, 2, $\cdots$, t-1) satisfy (5.24). Further, by Theorem 273 of [2], we get from (5.27) that

(5.31) $$d((p-2)!!) = \prod_{i=1}^{t-1}(a(i)+1).$$

Thus, by (5.26), we obtain (5.23). The theorem is proved.

References

[1] Hall, M., Combinatorial theory, London: Blaisdell Pub. Co., 1967.

[2] Hardy, G. H. and Wright, E. M., An introduction to the theory of numbers, Oxford: Oxford Univ. Press, 1938.

[3] Hua, L. –K., Introduction to number theory, Berlin: Springer-Verlag, 1982.

[4] Russo, F., An introduction to the Smarandache double factorial function, J. Smarandache Notions, 2001, 12: 158-167.

Department of Mathematics
Zhanjiang Normal College
Zhanjiang, Guangdong
P. R. CHINA