On Third Power Mean Values Computation of Digital Sum Function in Base n

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Abstract: let $m = a_1 n^{k_1} + a_2 n^{k_2} + \cdots\cdots + a_s n^{k_s}$, where $1 \le a_i < n$, $i = 1, 2, \cdots, s$. $k_1 > k_2 > \cdots > k_s \ge 0$, $a(m,n) = a_1 + a_2 + \cdots\cdots + a_s$, for $A_k(N,n) = \sum_{m<N} a^k(m,n)$ $(k = 1, 2, 3)$. An exact calculating formula for $A_k(N,n)$ $(k = 1,2,3)$ is given.

Key word: base n function of digital sum mean value

§1 Introduction and Main Results

In problem 21 of [1], Professor F.Smarandache asked us to study the properties of the sequences of digital sum. In paper [2] and [3] we give exact calculating formulas for $A_1(N,n)$ and $A_2(N,n)$. In this paper, we give an exact calculating formula for $A_3(N,n)$. For convenience, let

$$\varphi_k(n) = \sum_{i=1}^{n-1} i^k \quad , \qquad \varphi_1(n) = \frac{n(n-1)}{2} \quad , \qquad \varphi_2(n) = \frac{n(n-1)(2n-1)}{6} \quad .$$

First we have the following.

Definition. Assume n $(n \ge 2)$ be a fixed positive integer, for any positive integer m in base n, let $m = a_1 n^{k_1} + a_2 n^{k_2} + \cdots\cdots + a_s n^{k_s}$, where $k_1 > k_2 > \cdots > k_s \ge 0$, $1 \le a_i < n$, $i = 1, 2, \cdots, s$. Then

$a(m,n) = a_1 + a_2 + \cdots\cdots + a_s$ and for any positive integer r, $A_r(N,n) = \sum_{m<N} a^r(m,n)$.

Theorem 1. Let $N = a_1 n^{k_1} + a_2 n^{k_2} + \cdots\cdots + a_s n^{k_s}$, where $k_1 > k_2 > \cdots > k_s \ge 0$; $1 \le a_i < n$; $i = 1, 2, \cdots, s$, Then

$$\begin{aligned} &A_3(N,n) \\ &= \sum_{i=1}^{s} \Big(k_i a_i \varphi_1^2(n) \Big((2n-1) + \frac{1}{2}(n-1)(k_i - 3)k_i \Big) + 3\varphi_2(n) \big(2a_i \varphi_1(n)\varphi_1(k_i) + nk_i \varphi_1(a_i) \big) + 3n\varphi_1(n) \\ &\big((n-1)\varphi_1(a_i)\varphi_1(k_i) + k_i \varphi_2(a_i) \big) + n^2 \varphi_1^2(a_i) + 3n\Big(\sum_{j=1}^{i-1} a_j\Big)\Big(k_i a_i \varphi_2(n) + n\varphi_2(a_i) + (n-1)a_i \\ &\varphi_1(n)\varphi_1(k_i) + 2k_i \varphi_1(a_i)\varphi_1(n) \Big) + \frac{3}{2} n^2 a_i \Big(\sum_{j=1}^{i-1} a_j\Big)^2 \big((n-1)k_i + (a_i - 1) \big) + n^2 a_i \Big(\sum_{j=1}^{i-1} a_j\Big)^3 \Big) n^{k_i - 2} . \end{aligned}$$

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Corollary 1. Let $N=2^{k_1}+2^{k_2}+\cdots\cdots+2^{k_s}$, where $k_1>k_2>\cdots>k_s\ge 0$, then

$$A_3(N,2)=\sum_{i=1}^{s}\left(k_i^3+3(2i-1)k_i^2+6(i-1)(2i-1)k_i+8(i-1)^3\right)2^{k_i-3}.$$

Corollary 2. Let $N=a_1 10^{k_1}+a_2 10^{k_2}+\cdots\cdots+a_s 10^{k_s}$, where $1\le a_i<10$, $i=1,2,\cdots,10;$

$k_1>k_2>\cdots>k_s\ge 0$, then

$$A_3(N,10)=$$

$$25\sum_{i=1}^{s}\Big(40\varphi_1^2(a_i)+3645k_i(k_i-3)+180k_ia_i(a_i^2+8a_i+14)+15390k_i^2a_i-2430(k_i+a_i-1)+$$

$$30(\sum_{j=1}^{i-1}a_j)(36\varphi_1(a_i)+4\varphi_2(a_i)+3k_ia_i(27k_i+11))+60a_i(\sum_{j=1}^{i-1}a_j)^2(9k_i+a_i-1)+40(\sum_{j=1}^{i-1}a_j)^3\Big)10^{k_i-3}$$

§2 Proof of the theorem

In this section, we complete the proof of the theorem . First we have six simple lemmas. Let n, a are positive integers, k is an integer, we have five lemmas.

Lemma 1[2]. $A_1(n^k,n)=\dfrac{n-1}{2}kn^k.$ (1)

Lemma 2[2]. $A_1(an^k,n)=\dfrac{a}{2}((n-1)k+(a-1))n^k.$ (2)

Lemma 3[3]. $A_2(n^k,n)=(k\varphi_2(n)+(n-1)\varphi_1(n)\varphi_1(k))n^{k-1}.$ (3)

Lemma 4[3].

$$A_2(an^k,n)=(ka\varphi_2(n)+n\varphi_2(a)+(n-1)\varphi_1(n)\varphi_1(k)+2k\varphi_1(n)\varphi_1(a))n^{k-1}. \quad (4)$$

Lemma 5.

$$A_3(n^k,n)=\left(k\varphi_1^2(n)\left((2n-1)+\frac{1}{2}(n-1)(k-3)k\right)+6\varphi_1(n)\varphi_2(n)\varphi_1(k)\right)n^{k-2}. \quad (5)$$

Proof. We only prove the identity (5)

If $k=1$, then

The left of the equation $=A_3(n,n)=a^3(1,n)+(2-1,n)+\cdots\cdots+a^3(n-1,n)$

$$=1^3+2^3+\cdots\cdots+(n-1)^3$$

$$=\varphi_1^2(n).$$

The right of the equation $=n\varphi_1^2(n)\cdot n^{-1}=\varphi_1^2(n).$

So the left and right of the equation (5) equals, the proposition is correct .

Assume $k=p$, lemma (5) is correct. That is,

$$A_3(n^p,n)=\left(p\varphi_1^{\,2}(n)\left((2n-1)+\frac{1}{2}(n-1)(p-3)p\right)+6\varphi_1(n)\varphi_2(n)\varphi_1(p)\right)n^{p-2}.$$

Then $A_3(n^{p+1},n)=\sum_{m<n^{p+1}} a^3(m,n)$

$$=\sum_{m<n^p} a^3(m,n)+\sum_{n^p\le m<2n^p} a^3(m,n)+\cdots+\sum_{(n-1)n^p\le m<n^{p+1}} a^3(m,n)$$

$$=\sum_{m<n^p} a^3(m,n)+\sum_{0\le m<n^p}(a(m,n)+1)^3+\cdots+\sum_{0\le m<n^p}(a(m,n)+(n-1))^3$$

$$=n\sum_{m<n^p} a^3(m,n)+3\sum_{m<n^p} a^2(m,n)(\sum_{i=1}^{n-1} i)+3\sum_{m<n^p} a(m,n)(\sum_{i=1}^{n-1} i^2)+(\sum_{i=1}^{n-1} i^3)n^p$$

$$=nA_3(n^p,n)+3\varphi(n)A_2(n^p,n)+3\varphi_2(n)A_1(n^p,n)+\varphi_1^{\,2}(n)n^p.$$

Combining inductive assume, (1) and (2), we immediately get

$$A_3(n^{p+1},n)=\left(p\varphi_1^{\,2}(n)\left((2n-1)+\frac{1}{2}(n-1)(p-3)p\right)+6\varphi_1(n)\varphi_2(n)\varphi_1(p)\right)n^{p-1}+3\varphi_1(n)\left(p\varphi_2(n)\right.$$

$$\left.+\varphi_1(n)\varphi_1(p)(n-1)\right)n^{p-1}+\frac{3}{2}(n-1)p\varphi_2(n)n^p+\varphi_1^{\,2}(n)n^p$$

$$=\left((p+1)\varphi_1^{\,2}(n)\left((2n-1)+\frac{1}{2}(n-1)(p-2)(p+1)\right)+6\varphi_1(n)\varphi_2(n)\varphi_1(p+1)\right)n^{p-2}.$$

So lemma 5 is also correct for $k=p+1$.

Lemma 6.

$$A_3(an^k,n)=\left(ka\varphi_1^{\,2}(n)\left((2n-1)+\frac{1}{2}(n-1)(k-3)k\right)+3\varphi_2(n)\left(2a\varphi_1(n)\varphi_1(k)+kn\varphi_1(a)\right)\right.$$

$$\left.+3n\varphi_1(n)\left((n-1)\varphi_1(a)\varphi_1(k)+k\varphi_2(a)\right)+n^2\varphi_1^{\,2}(a)\right)n^{k-2} \qquad (6)$$

proof. $A_3(an^k,n)=\sum'_{m<an^k} a^3(m,n)$

$$=\sum_{m<n^k} a^3(m,n)+\sum_{n^k\le m<2n^k} a^3(m,n)+\cdots+\sum_{(a-1)n^k\le m<an^k} a^3(m,n)$$

$$=\sum_{m<n^k} a^3(m,n)+\sum_{0\le m<n^k}(a(m,n)+1)^3+\cdots+\sum_{0\le m<n^k}(a(m,n)+(a-1))^3$$

$$=a\sum_{m<n^k} a^3(m,n)+3\sum_{m<n^k} a^2(m,n)(\sum_{i=1}^{a-1} i)+3\sum_{m<n^k} a(m,n)(\sum_{i=1}^{a-1} i^2)+(\sum_{i=1}^{a-1} i^3)n^k$$

$$=aA_3(n^k,n)+3\varphi(a)A_2(n^k,n)+3\varphi_2(a)A_1(n^k,n)+\varphi_1^{\,2}(a)n^k$$

Combining (1), (3) and (5), we get

$$A_3(an^k,n)=\left(ka\varphi_1^{\,2}(n)\left((2n-1)+\frac{1}{2}(n-1)(k-3)k\right)+3\varphi_2(n)\left(2a\varphi_1(n)\varphi_1(k)+kn\varphi_1(a)\right)\right.$$

$$\left.+3n\varphi_1(n)\left((n-1)\varphi_1(a)\varphi_1(k)+k\varphi_2(a)\right)+n^2\varphi_1^{\,2}(a)\right)n^{k-2}.$$

This proves lemma 6.

Now we use the above six lemmas to complete the proof of the theorem,

$$A_3(N,n) = \sum_{m<N} a^3(m,n)$$

$$= \sum_{m<a_1 n^{k_1}} a^3(m,n) + \sum_{a_1 n^{k_1} \le m < a_1 n^{k_1} + a_2 n^{k_2}} a^3(m,n) + \cdots + \sum_{N - a_s n^{k_s} \le m < N} a^3(m,n)$$

$$= \sum_{m<a_1 n^{k_1}} a^3(m,n) + \sum_{0 \le m < a_2 n^{k_2}} (a(m,n)+a_1)^3 + \cdots + \sum_{0 \le m < a_s n^{k_s}} (a(m,n) + \sum_{i=1}^{s-1} a_i)^3$$

$$= \sum_{i=1}^{s} A_3(a_i n^{k_i}) + 3\sum_{i=1}^{s}(\sum_{j=1}^{i-1} a_j) A_2(a_i n^{k_i}) + 3\sum_{i=1}^{s}(\sum_{j=1}^{i-1} a_j)^2 A_1(a_i n^{k_i}) + \sum_{i=1}^{s}(\sum_{j=1}^{i-1} a_j)^3 a_i n^{k_i}$$

From (2), (4) and (6), we have

$A_3(N,n)$

$$= \sum_{i=1}^{s} \Big(k_i a_i \varphi_1{}^2(n)\big((2n-1) + \frac{1}{2}(n-1)(k_i-3)k_i\big) + 3\varphi_2(n)\big(2a_i\varphi_1(n)\varphi_1(k_i) + nk_i\varphi_1(a_i)\big) + 3n\varphi_1(n)$$

$$\big((n-1)\varphi_1(a_i)\varphi_1(k_i) + k_i\varphi_2(a_i)\big) + n^2\varphi_1{}^2(a_i) + 3n(\sum_{j=1}^{i-1} a_j)\big(k_i a_i \varphi_2(n) + n\varphi_2(a_i) + (n-1)a_i$$

$$\varphi_1(n)\varphi_1(k_i) + 2k_i\varphi_1(a_i)\varphi_1(n)\big) + \frac{3}{2}n^2 a_i (\sum_{j=1}^{i-1} a_j)^2\big((n-1)k_i + (a_i-1)\big) + n^2 a_i (\sum_{j=1}^{i-1} a_j)^3\Big) n^{k_i-2}$$

This completes the proof of the Theorem.

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