ON THREE NUMERICAL FUNCTIONS

by

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In this paper we define the numerical functions φ_S, φ_S^*, ω_S and we prove some properties of these functions.

1. Definition. If S(n) is the Smarandache function, and (m, n) is the greatest common divisor of m and n, then the functions φ_S, φ_S^* and ω_S are defined on the set **N*** of the positive integers, with values in the set N of all the non negative integers, such that:

$\varphi_S(x) = \text{Card}\{m \in \mathbf{N}^* \,/\, 0 < m \le x, (S(m), x) = 1\}$

$\varphi_S{}^*(x) = \text{Card}\{m \in \mathbf{N}^* \,/\, 0 < m \le x, (S(m), x) \ne 1\}$

$\omega_S(x) = \text{Card}\{m \in \mathbf{N}^* \,/\, 0 < m \le x, \text{ and } S(m) \text{ divides } x\}$.

From this definition it results that:

$$\varphi_S(x) + \varphi_S{}^*(x) = x \text{ and } \omega_S(x) \le \varphi_S{}^*(x) \tag{1}$$

for all $x \in \mathbf{N}^*$.

2. Proposition. For every prime number $p \in \mathbf{N}^*$ we have

$\varphi_S(p) = p - 1 = \varphi(p)$, $\varphi_S(p^2) = p^2 - p = \varphi(p^2)$

where φ is Euler's totient function.

Proof. Of course, if p is a prime then for all integer a satisfying $0 < a \le p - 1$ we have $(S(a), p) = 1$, because $S(a) \le a$. So, if we note $M_1(x) = \{m \in \mathbf{N}^* \,/\, 0 < m \le p, (S(m), p) = 1\}$ then $a \in M_1(p)$.

At the same time, because $S(p) = p$, it results that $(S(p), p) = p \ne 1$ and so $p \notin M_1(p)$.

Then we have $\varphi_S(p) = p - 1 = \varphi(p)$.

The positive integers a, not greater than p^2 and not belonging to the set $M_1(p^2)$ are: $p, 2p, \dots, (p-1)p, p^2$.

For $p = 2$ this assertion is evidently true, and if p is an odd prime number then for all $h < p$ it results $S(h \cdot p) = p$.

Now, if $m < p^2$ and $m \neq hp$ then $(S(m), p^2) = 1$. Indeed, if for $m = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, $q_i \neq p$ we have $(S(m), p^2) \neq 1$, then it exists a divisor q^α of m such that $S(m) = S(q^\alpha) = q(\alpha - i_\alpha)$, with $i_\alpha \in \left[0; \left[\frac{\alpha - 1}{q}\right]\right]$.

From $(q(\alpha - i_\alpha), p^2) \neq 1$ it results $(q(\alpha - i_\alpha), p) \neq 1$ and because $q \neq p$ it results $(\alpha - i_\alpha, p) \neq 1$, so $(\alpha - i_\alpha, p) = p$. But p does not divide $\alpha - i_\alpha$ because $\alpha < p$.

Indeed, we have:

$$q^\alpha < p^2 \Leftrightarrow \alpha < 2\log_q p \leq 2 \cdot \frac{p}{2} = p$$

because we have:

$$\log_q p \leq \frac{p}{2} \text{ for } q \geq 2 \text{ and } p \geq 3.$$

So,

$$\varphi_S(p^2) = p^2 - \text{Card}\{1 \cdot p, 2 \cdot p, \dots, (p-1)p, p^2\} = p^2 - p = \varphi(p^2).$$

3. Proposition. For every $x \in \mathbf{N^*}$ we have:

$$\varphi_S(x) \leq x - \tau(x) + 1$$

where $\tau(x)$ is the number of the divisors of x.

Proof. From (1) it results that $\varphi_S(x) = x - \varphi_S{}^*(x)$, and of course, from the definition of $\varphi_S{}^*$ and τ it results $\varphi_S{}^*(x) \geq \tau(x) - 1$. Then $\varphi_S(x) \leq x - \tau(x) + 1$. Particularly, if x is a prime then $\varphi_S(x) \leq x - 1$, because in this case $\tau(x) = 2$.

If x is a composite number, it results that $\varphi_S(x) \leq x - 2$.

4. Proposition. If $p < q$ are two consecutive primes then :

$$\varphi_S(pq) = \varphi(pq).$$

Proof. Evidently, $\varphi(pq) = (p-1)(q-1)$ and

$$\varphi_S(pq) = \text{Card}\{m \in \mathbf{N^*} \,/\, 0 < m \leq pq, (S(m), pq) = 1\}.$$

Because p and q are consecutive primes and $p < q$ it results that the multiples of p and q which are not greater than pq are exactly given by the set:

$$M = \{p, 2p, \dots, p^2, (p+1)p, \dots, (q-1)p, qp, q, 2q, \dots, (p-1)q\}.$$

These are in number of $p + q - 1$.

Evidently, $(S(m), pq) \neq 1$ for $m \in \{p, 2p, \ldots, (p-1)p, p^2, q, 2q, \ldots, (p-1)q\}$.

Let us calculate $S(m)$ for $m \in \{(p+1)p, (p+2)p, \ldots, (q-1)p\}$.

Evidently, $(p+i, p) = 1$ for $1 \leq i \leq q - p - 1$, and so $[p+i, p] = p(p+i)$.

It results that $S(p(p+i) = S([p, p+i]) = \max\{S(p), S(p+i)\} = S(p)$.
Indeed, to estimate $S(p+i)$ let $p + i = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_h^{\alpha_h} < q < 2p$.
Then $p_1^{\alpha_1} < p, p_2^{\alpha_2} < p, \ldots, p_h^{\alpha_h} < p$.
It results that:
$S(p+i) = S(p_j^{\alpha_j}) < S(p)$, for some $j = \overline{1,h}$.
It results that:

$(S(p(p+i), pq) = (p, pq) = p \neq 1$.

In the following we shall prove that if $0 < m \leq pq$ and m is not a multiple of p or q then $(S(m), pq) = 1$.

It is said that if $m < p^2$ is not a multiple of p then $(S(m), p) = 1$.

If $m \leq q^2$ is not a multiple of q then it results also $(S(m), q) = 1$.

Now, if $m < p^2$ (and of course $m < q^2$) is not a multiple either of p and q then from $(S(m), p) = 1$ and $(S(m), q) = 1$ it results $(S(m), pq) = 1$.

Finally, for $p^2 < m < pq < q^2$, with m not a multiple either of p and q, if the decomposition of m into primes is $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_S^{\alpha_S}$ then $S(m) = S(p_k^{\alpha_k}) < S(p) = p$ so $(S(m), p) = 1$.

Analogously, $(S(m), q) = 1$, and so $(S(m), pq) = 1$.

Consequently,

$$\varphi_S(pq) = pq - p - q + 1 = \varphi_S(pq).$$

5. Proposition

(i) If $p > 2$ is a prime number then $\omega_S(p) = 2$, $\omega_S(p^2) = p$.

(ii) If x is a composite number then $\omega_S(x) \geq 3$.

Proof. From the definition of the function ω_S it results that $\omega_S(p) = 2$.

If $1 \le m \le p^2$, from the condition that $S(m)$ divides p^2 it results $m = 1$ or $m = kp$, with

$k \le p - 1$, so :

$$m \in \{1, p, 2p, \ldots, (p-1)p\} \quad \text{and} \quad \omega_S(p^2) = p.$$

If x is a composite number, let p be one of its prime divisors.

Then, of course, $1, p, 2p \in \{m / 0 < m \le x\}$.

If $p > 3$ then :

$S(1) = 1$ divides x, $S(p) = p$ divides x and $S(2p) = S(p) = p$ divides x.

It rezults $\omega_S(x) \ge 3$.

If $x = 2^\alpha$, with $\alpha \ge 2$ then :

$S(1) = 1$ divides x, $S(2) = 2$ divides x and $S(4) = 4$ divides x ,

so we have also $\omega_S(x) \ge 3$.

6. Proposition. For every positive integer x we have :

$$\omega_S(x) \le x - \varphi(x) + 1. \tag{2}$$

Proof. We have $\varphi(x) = x - \text{Card } A$, when

$$A = \{m / 0 < m \le x, (m, x) \ne 1\}.$$

Evidently, the inequality (2) is valid for all the prime numbers.

If x is a composite number it results that at least a proper divisor of m is also a divisor of $S(m)$ and of x. So $(m, x) \ne 1$ and consequently $m \in A$.

So, $\{m / 0 < m \le x, S(m) \text{ divides } x\} \subset A \cup \{1\}$ and it results that :

$\text{Card } \{m / 0 < m \le x, S(m) \text{ divides } x\} \le \text{Card } A - 1$, or

$\omega_S(x) \le 1 + \text{Card } A$,

and from this it results (2).

7. Proposition. The equation $\omega_S(x) = \omega_S(x + 1)$ has not a solution between the prime numbers.

Proof. Indeed, if x is a prime then $\omega_S(x) = 2$ and because $x + 1$ is a composite number it results $\omega_S(x + 1) \ge 3$.

Let us observe that the above equation has solutions between the primes. For instance, $\omega_S(35) = \omega_S(36) = 11$.

8. Proposition. The function $\varphi_S(x)$ has all the primes as local maximal points.

Proof. We have $\varphi_S(p) = p - 1$, $\varphi_S(p - 1) \leq p - 3 < \varphi_S(p)$ and $\varphi_S(p + 1) \leq \varphi_S(p)$, because $p + 1$ being a composite number has at least two divisors.

Let us mention now the following unsolved problems:

(UP$_1$) There exists $x \in \mathbf{N}^*$ such that $\varphi_S(x) < \varphi(x)$.

(UP$_2$) For all $x \in \mathbf{N}^*$ is valid the inequality

$$\omega_S(x) \geq \tau(x)$$

where $\tau(x)$ is the number of the divisors of x.

References

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