PAPER MODELS OF SURFACES WITH CURVATURE
CREATIVE VISUALIZATION LABS
BALTIMORE JOINT MATHEMATICS MEETINGS

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ABSTRACT. A model of a cone can be constructed from a piece of paper by removing a wedge and taping the edges together. The paper models discussed here expand on this idea (one or more wedges are added and/or removed). These models are flat everywhere, except at the "cone points," so the geodesics are locally straight lines in a natural sense. Non-Euclidean "effects" are easily quantifiable using basic geometry, the Gauss-Bonnet theorem is a naturally intuitive concept, and the connection between hyperbolic and elliptic geometry and curvature is clearly seen.

1. OBJECTIVES AND NOTES

The notion that a geometric space can be manipulated is an idea that I would like to instill in students. A number of behaviors of lines/geodesics can be found by constructing a variety of surfaces. I believe that this can be of value, as it is in topology where metric spaces with marginally intuitive properties are readily available. All of the models described in the labs are essentially 2-manifolds, so the notion that there are many accessible manifolds will hopefully be carried by the student into a study of differential geometry or topology.

The local geometry of these paper models corresponds directly to the geometry of smoothly curved surfaces, so they can be used as an introduction to a study of Riemannian geometry. Geodesics on these surfaces are easy to find, since they are straight lines when the paper is flattened, and a protractor can measure the *angle defect*, which is essentially equivalent to a measure of total curvature. Since the Gauss curvature is an infinitesimal version of the angle defect, the definition of Gauss curvature can be motivated in terms of these models. Furthermore, there is a polyhedral version of the Gauss-Bonnet theorem that is easy to see, and this can be used to make sense of the smooth version.

These labs come from a series of projects I gave to three students doing an independent study course in geometry. The three worked together on these projects with very little help from me, and while these students were stronger than average, I think the labs are appropriate for outside-of-class assignments that are independent of the main course of study. I would assign one lab a week in the month prior to starting non-Euclidean geometry.

2. INTRODUCTION

The geometry of a sphere is fundamentally different from that of the plane. The essence of this difference is captured in the Gauss curvature, where the sphere has constant positive curvature and the plane has zero curvature everywhere. This difference in curvature and geometry manifests itself in the inability to build paper models of the sphere out of flat pieces of paper. A cylinder, on the other hand, is easily constructed from paper, and correspondingly has the same (Gauss) curvature and local geometry as the plane. In fact, the geodesics on the cylinder correspond to straight lines on the paper when flat, and a cylindrical paper model quickly leads to the realization that the cylindrical geodesics are helixes (degenerate and non-degenerate).

A cone can also be constructed from paper. The geodesics, while not as easily described as for the cylinder, can be seen the same way. One characteristic that the cone and sphere share is that no region containing the vertex can be flattened (without tearing the paper). The cone and sphere also share a notion of positive curvature and an elliptic geometry.

The cone formed by removing a wedge measuring θ radians is defined to have an *angle defect* equal to θ. I prefer the term *impulse curvature,* since the angle defect corresponds to a Gauss curvature singularity at the cone point with a finite integral. In fact, if you were to smoothly round off the vertex of the cone and integrate the Gauss curvature, you would get a total curvature of precisely θ. As a result, the Gauss-Bonnet theorem extends nicely to angle defects. Actually, the Gauss-Bonnet theorem on a cone is obvious once you know what to look for, and perhaps we should say that the Gauss-Bonnet theorem is an extension of a polyhedral version due to Descartes. All of this applies equally well to hyperbolic geometry, since adding a wedge introduces a negative angle defect and a negative total curvature.

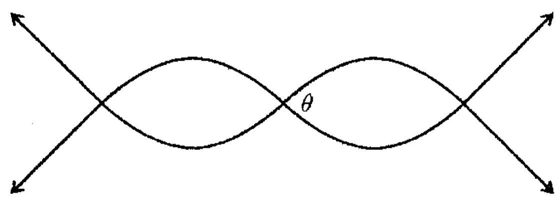

FIGURE 1. A pair of geodesics with three points of intersection

3. A SAMPLE PROBLEM FROM LAB 2

One of the problems in Lab 2 asks the students to construct a surface that has a pair of geodesics with three points of intersection. If the geodesics are to be configured as in Figure 1, they will form two regions bounded by 2-gons. The Gauss-Bonnet theorem requires that the total curvature in each region must equal the angle sum of its bounding 2-gon. If we want the angle at the middle intesection point to be θ radians, therefore, then we need to introduce total curvature greater than θ inside each region. In terms of cone points, we need to introduce two cone points by removing wedges that measure more than θ radians.

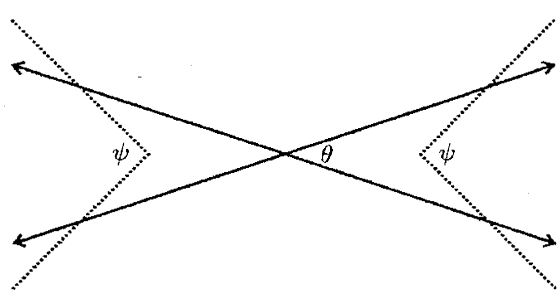

FIGURE 2. We can remove wedges measuring more than θ radians.

We can construct the surface as follows. Start with two lines intersecting at a single point as in Figure 2. We can make the pair of lines intersect twice more by removing two wedges, and clearly

ψ must be greater than θ (without using the Gauss-Bonnet theorem at all). The result is two cone points each with total curvature ψ.

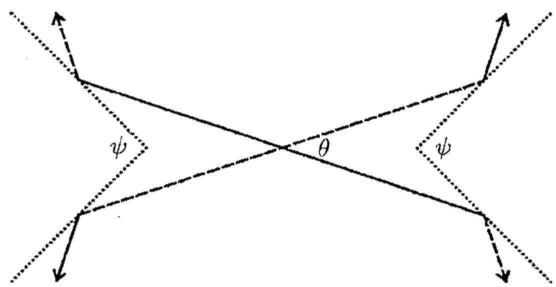

FIGURE 3. The continuations of the geodesics on the surface will look like this.

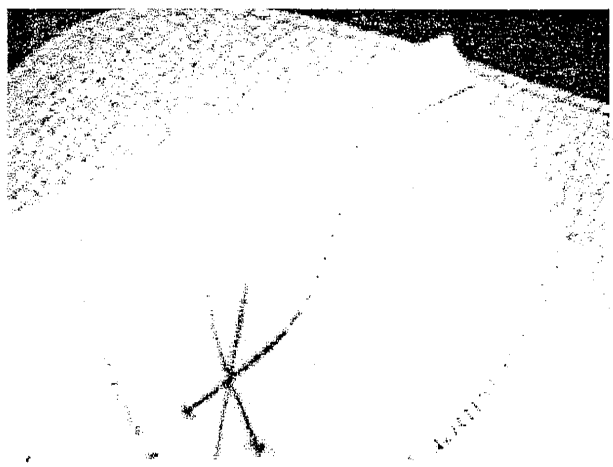

FIGURE 4. The paper model corresponding to Figure 3 looks like this.

The particular values for θ and ψ can vary greatly, but $\theta = 45°$ and $\psi = 90°$ is convenient to draw, and a paper model can be constructed from the diagram in Figure 3. I have drawn one geodesic solid and one broken to distinguish them. Note that the continuations of each geodesic must intersect the cut at the same angle, so it's easy to do with a ruler and protractor, if you choose convenient angles. The resulting paper model is shown in Figure 4.

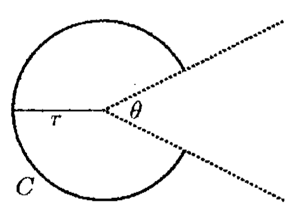

FIGURE 5. The angle defect corresponds to total curvature.

4. Gauss-Bonnet Theorem

I do not address the Gauss-Bonnet theorem in any of the labs, but after the students have completed the last lab, I would look at the cone point version of the Gauss-Bonnet theorem. From here, the definition for Gauss curvature on a smooth surface should make sense intuitively.

The basic idea can be seen using circles and spheres. Consider a circle of radius r centered at the cone point of a cone with angle defect θ, as in Figure 5. In the plane, this circle will have curvature $\kappa = \frac{1}{r}$. Since the local geometry on the cone is Euclidean away from the cone point, the *geodesic* curvature for this circle as a curve on the cone must be the same. That is, $\kappa_g = \frac{1}{r}$. What is different about this circle and a circle in the plane with the same radius, is that the circle on the cone has a smaller circumference. In fact, the difference must be θr.

We can now compute the total geodesic curvature.

$$\int_C \kappa_g \, ds = \frac{1}{r} \int_C ds = \frac{1}{r}(2\pi r \quad \theta r) = 2\pi \quad \theta. \tag{1}$$

Since curvature measures the rate of rotation of the tangent vector, it should make sense to students that the total rotation for a simple closed curve in the plane must always be 2π. Since any small deformation of the circle essentially takes place in the plane, it should also make sense that the total rotation for a simple closed curve around the cone point will always be 2π minus the angle defect. In any case, the formulation of the Gauss-Bonnet theorem should seem natural.

Comparing Equation (1) to the Gauss-Bonnet theorem,

$$\int_C \kappa_g \, ds = 2\pi \quad \int_R K \, dA, \tag{2}$$

it's obvious that the angle defect corresponds with the total curvature $\int K \, dA$. In fact, I think it makes perfect sense to motivate the definition of the Gauss curvature K in terms of this formula. I might start out by doing the following.

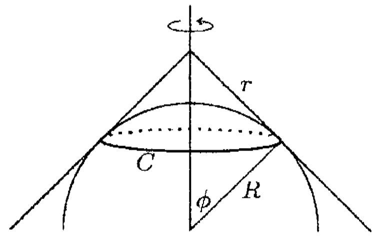

FIGURE 6. The circle of tangency will have the same geodesic curvature on both surfaces.

Consider a sphere tangent to a cone, as shown in Figure 6. The geodesic curvature for the circle of tangency will be the same on both surfaces. Therefore, the total curvature for the regions contained by the circle on both surfaces should be the same. We can then require that the Gauss curvature be an infinitesimal version of the total curvature and that it be constant on the sphere. That is,

$$\theta = \int_D K \, dA = K \int_D dA = K R^2 \theta, \tag{3}$$

and

$$K = \frac{1}{R^2}. \tag{4}$$

I think the actual computation is a bit tricky, but there may be a simpler way. In any case, the area integral is

$$\int_D dA = \int_0^{2\pi} \int_0^{\phi} R^2 \sin p \, dp dt = R^2(1 \quad \cos\phi)2\pi, \tag{5}$$

where the parameters p and t are the phi and theta from spherical coordinates. To express this expression in terms of θ, note that the circumference of the circle C is $2\pi r \quad \theta r$ on the cone. If the radius of this circle in space is ρ, then this circumference is also $2\pi\rho$. Since $R \sin\phi = \rho$, we have that

$$2\pi r \quad \theta r = 2\pi R \sin\phi, \tag{6}$$

and

$$\theta = 2\pi(1 \quad \frac{R}{r} \sin\phi). \tag{7}$$

Now, $\tan\phi = \frac{r}{R}$, so

$$\theta = 2\pi(1 \quad \frac{\cos\phi}{\sin\phi} \sin\phi) = 2\pi(1 \quad \cos\phi). \tag{8}$$

Equations (5) and (8) establish equation (3).

5. Further Reading

Total curvature was studied at least as far back as Descartes, where he used the term *inclination of the solid angle* in his investigations of convex polyhedra. It seems that the term *angle defect* is now standard. As mentioned, Descartes also had a formula that is a Gauss-Bonnet theorem for convex polyhedra. I've found some historical bits about this in [**1**], but I'm not sure if Gauss knew about Descartes' work when he was studying the curvature of surfaces. I intend to check this out eventually, but I get the sense that the geometry of cone points is too obvious to mention for working geometers, so this may have been the case for Gauss as well.

I first became aware of Descartes' work with angle defects from an article by H. Gottlieb called "All the way with Gauss-Bonnet" in the *Math Monthly* [**2**], and an article on the AMS website called "Descartes's lost theorem" [**4**]. The first article is an excellent second introduction to curvature.

My general interest started through my involvement with the Smarandache Geometry Club (Yahoo). The members of this club were interested in geometric spaces that satisfied Euclidean axioms in some instances and violated them in others. This would be somewhat normal in a Riemannian manifold, and I remembered reading about something Jeff Weeks called *hyperbolic paper* in *The Shape of Space* ([**6**]). This hyperbolic paper was constructed by taping equilateral triangles together so that there were seven triangles around each vertex. The result is a paper model with a bunch of cone points with angle defect equal to $\quad \frac{\pi}{3}$. Building on this idea, I was able to build a lot of models that exhibited properties that the members of the club were looking for, and I eventually wrote a little book on the subject called *Smarandache Manifolds* ([**3**]). I think one of the difficulties in motivating proofs in Euclidean geometry is that students have a hard time imagining how any of the theorems could not be true. It's hard to justify a confusing proof for a statement that is obviously true. This book has lots of counter-examples. I have copies to give away, so let me know, if you want one.

I think cone points come up in the study of orbifolds, but they seem to fit most naturally in an area called computational geometry. I know almost nothing about either of these subjects, but [**5**] is a nice, accessible article by two leading computational geometers.

References

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(3) H. Iseri, Smarandache Manifolds, American Research Press, Rehoboth, NM, USA, 2002. (available at www.Gallup.unm.edu/~smarandache/Iseri-book.PDF)

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(5) K. Polthier and M. Schmies, Straightest geodesics on polyhedral surfaces, Mathematical Visualizations, 1998.

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